MATH 54 - LECTURE 6: CLOSED SETS, INTERIOR, BOUNDARY, LIMIT POINTS

DAN CRYTSER

Introduction

In this lecture we introduce the notion of a closed set in a topological space. This allows to define the boundary of a set, and we also define the interior of a set. We define limit points of sets.

Closed Sets

Closed sets are intended to generalize the familiar closed intervals from the real line: $[a, b]$. You can also think of them as generalizing the closed ball $\overline{B}(x, \epsilon) = \{y \in \mathbb{R}^n : ||x - y|| \leq \epsilon\}$ in Euclidean space.

Definition Let X be a topological space. Then a subset $A \subset X$ is closed if $X \setminus A = \{x \in X : x \notin A\}$ is open in X.

Example The closed interval $A = [0, 1]$ is a closed subset of $\mathbb{R}$ (with the std topology). For the complement $\mathbb{R} \setminus A = (-\infty, 0) \cup (1, \infty)$ is open, as you can check. The same applies to any other closed interval $[a, b]$ with $a \leq b$.

Example Let X be a set. The cofinite topology has as its open sets all sets with finite complement, as well as the null set $\emptyset$. Thus the complements of the open sets are the finite sets, as well as the total set X. So the closed sets in the finite complement topology are the finite subsets of X as well as X. Notice that in the integers, equipped with the cofinite topology, the collection of even integers is neither open nor closed.

Example In the discrete topology on a set X, every subset of X is open, hence every subset is closed. So in the discrete topology every subset of X is both closed and open. In the indiscrete topology on X, the open subsets are X and $\emptyset$, hence these are the closed sets also. Notice that in the indiscrete topology one-point sets are not closed unless X contains only one point.

Example Let $X = \mathbb{R}$ and $Y = [0, 1] \cup (2, 3)$. We can endow Y with the subspace topology coming from X. The subset $(2, 3) \subset Y$ is open in Y, as $(2, 3) = (2, 3) \cap Y$, and $(2, 3)$ is open as a subset of $\mathbb{R}$. Similarly, $[0, 1]$ is open as a subset of Y, for $[0, 1] = (-1, 2) \cap Y$, and $(-1, 2)$ is open in $\mathbb{R}$. Thus $[0, 1]$ and $(2, 3)$ are both open and closed. This is a good example of a disconnected space – one in which there are non-trivial (not X or $\emptyset$) subsets which are both closed and open.

The deMorgan laws allow us to translate the axioms for open subsets of a topological space into statements about closed subsets:

Theorem 1. *Let X be a topological space. Then the following statements are true:*

(1) *$\emptyset$ and X are closed subsets of X*

(2) *If $\{F_i\}_{i\in I}$ is a collection of closed subsets of X, then $\cap_{i\in I} F_i$ is also closed in X. That is, arbitrary intersections of closed sets are closed.*
(3) *If $F_1, F_2, \ldots, F_n$ is a finite collection of closed subsets of X, then $F_1 \cup F_2 \ldots \cup F_n$ is also closed in X. That is, finite unions of closed sets are closed.*

Proof. (1) As X is the complement of $\emptyset$ and vice versa, and both sets are open, we see that both sets are also closed.
(2) The complement of $\cap_{i\in I} F_i$ is $\cup_{i\in I} X \setminus F_i$, by deMorgan's laws, and the latter union is open because each set $X \setminus F_i$ is open. Thus the complement of $\cap_{i\in I} F_i$ is open, and so $\cap_{i\in I} F_i$ must be closed.
(3) The complement of $\cup_{i=1}^{n} F_i$ is $\cap_{i=1}^{n} X \setminus F_i$, and the latter intersection is open as it is the finite intersection of open sets. Thus the complement of $\cup_{i=1}^{n} F_i$ is open, and so this union must be closed.

□

Recall that if X is a top. space and $Y \subset X$, then $A \subset Y$ is open in the subspace topology on Y if there exists $U \subset X$, open in X, which satisfies $U \cap Y = A$. The following theorem allows us to recognize closed sets in the subspace topology:

Theorem 2. *Let Y be a subspace of X (i.e. X is a topological space and $Y \subset X$ is given the subspace topology). Then $A \subset Y$ is closed in Y if and only if there is a closed subset $C \subset X$ such that $C \cap Y = A$.*

Proof. Suppose that $A = C \cap Y$, where $C \subset X$ is closed. Then $Y \setminus A = (X \setminus C) \cap Y$, and this latter set is open in the subspace topology. Thus the complement of A in Y is open in Y, and so A is closed.

Suppose that $A \subset Y$ is closed in the subspace topology. Then $Y \setminus A = U \cap Y$ for some open subset U of X. But then $A = (X \setminus U) \cap Y$, and $X \setminus U$ is closed in X as $U \subset X$ is open. □

Theorem 3. *Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X.*

Proof. If A is closed in Y, it equals the intersection of Y with a closed subset $C \subset X$. But then $A = Y \cap C$ is an intersection of two closed sets in X, and so it must be closed itself. □

Boundary, Closure, and Interior

Let $A \subset X$. Then the collection of all closed subsets F such that $A \subset F \subset X$ is non-empty (take $X = F$ for example). Thus we can take the intersection of all closed subsets $F \subset X$ such that $A \subset F$. This intersection is called the closure of A, and is denoted $\overline{A}$. The union of all open sets $U \subset A$ is called the interior of A. Note that the interior is open, as it is the union of open sets. We denote the interior by int A or A°, and we won't spend too much time on it.

Definition If $x \in X$ and $U \subset X$ is an open subset of X containing x, then we call U a neighborhood or open neighborhood of x (some mathematicians extend the idea of nbhd of x to any set which contains an open neighborhood of x – in this definition $[0, 1]$ is a neighborhood of $1/2$. We will not use this definition.)

Theorem 4. *Let X be a space, let $A \subset X$ and $x \in X$. Then $x \in \overline{A}$ if and only if every neighborhood about x contains a point of A.*

Proof. We prove that $x \notin \overline{A}$ if and only if there is a neighborhood of x that does not contain a point of A. If $x \notin \overline{A}$, there is a closed subset $F \subset X$ with $A \subset F$ and $x \notin F$. Then $X \setminus F$ is an open neighborhood of x not intersecting A. If U is such a neighborhood, $X \setminus U$ can be taken as F in the previous paragraph. □

Limit Points

The point 0 is the limit of the infinite sequence $a_n = \frac{1}{n}$ in $\mathbb{R}$. This is an example of a limit point of a set: **Definition** Let X be a top. space, let $A \subset X$. Then $x \in X$ is a limit point of A if every nbhd of x contains a point of A which is distinct from x.

Example The point 0 of $\mathbb{R}$ is a limit point of $(0, 1)$ (if $\mathbb{R}$ has std top) because every open ball around 0 contains a point of $(0, 1)$.

Example The point 0 of $\mathbb{R}_\ell$ is not a limit point of $A = (-1, 0]$, because the nbhd $[0, 1)$ of 0 does not contain a point of A different from 0.

Example Let X be an infinite set, let $A \subset X$ be infinite. Then every point $x \in X$ is a limit point of A, because a neighborhood of x excludes at most finitely many points of A.

Example Each of the integers $n \in \mathbb{N}$ fails to be a limit point of any subset of $\mathbb{N}$, as the one-point nbhd $\{n\}$ cannot intersect any set in a point different from n. This is true in general for discrete spaces.

Definition If $A \subset X$, let A' denote the collection of limit points of A (which may be empty).

Theorem 5. *Let $A \subset X$. Then $\overline{A} = A \cup A'$.*

Proof. If $x \in \overline{A}$, then every neighborhood of x contains a point of A. If $x \notin A$, then this is always a point of A distinct from x, and so $x \in A' \subset A \cup A'$. If $x \in A$, then $x \in A \cup A'$. Thus $\overline{A} \subset A \cup A'$. If $x \in A \cup A'$, then let U be a neighborhood of x. Either $x \in A$, so that $A \cap U \neq \emptyset$, or $x \in A'$, in which case U contains a point of A distinct from x. In both cases the neighborhood U contains a point of A, and so $x \in \overline{A}$. □

Thus we can view the closure operation as "adding in the limit points."

Convergence

Convergence in Euclidean space – discuss. Sequence enters and stays within every open ball about a.

Let $(x_n)_{n=1}^{\infty}$ be a sequence of points in a topological space X, and let $a \in X$. We say that x_n converges to a if for every open nbhd U of a there exists some integer N such that if $n > N$, we have $x_n \in U$.

Example The sequence $a_n = 1/n$ converges to 0 in the standard topology on $\mathbb{R}$ **Example** The sequence $a_n = 1/n$ converges to every point of $\mathbb{R}$ in the indiscrete topology. **Example** The sequence $a_n = 1/n$ does not converge to any point of $\mathbb{R}$ in the discrete topology. **Example** The sequence $a_n = 1/n$ converges to 0 in $\mathbb{R}_\ell$. **Example** The sequence $b_n = -1/n$ does not converge to any point of $\mathbb{R}_\ell$. **Example** The sequence $a_n = 1/n$ does not converge to any point of $\mathbb{R}_K$. **Example** The sequence $a_n = 1/n$ converges to 0 in the cofinite topology on $\mathbb{R}$. **Example** The sequence $a_n = 1/n \times 1/n$ does not converge to 0 in $\mathbb{R}^2$ equipped with the order topology, as the open neighborhood $0 \times (-1, 1)$ never contains any points of this sequence. The sequence $a_n = (b_n, c_n)$ converges to $a = (b, c)$ iff $b_n = b$ for all but finitely many n and if $c_n \to c$ in the standard topology on $\mathbb{R}$.

Hausdorff Spaces

In $\mathbb{R}$ the singletons $\{x\}$ are all closed sets. For the complement is $(-\infty, x) \cup (x, \infty)$, which is open.

In a general topological space X, the singletons need not be closed. For example if we endow $\mathbb{R}$ with the concrete topology, which has as its closed sets $\mathbb{R}$ and $\emptyset$, none of the singletons are closed. Another example takes $X = \{a, b\}$ and $\tau = \{X, \emptyset, \{a\}\}$. Then the singleton $\{a\}$ is not closed, as the complement $\{b\}$ is not open.

Another upsetting thing about this space is that the constant sequence $x_n = a$ converges to a as well as b, whereas the constant sequence $y_n = b$ only converges to b.

Definition A top. space X is Hausdorff if for every $x, y \in X$ with $x \neq y$, there exist open nbhds U and V of x and y respectively such that $U \cap V = \emptyset$.

Example Euclidean space $\mathbb{R}^n$ is Hausdorff. Let $x, y \in \mathbb{R}^n$, and let $d = ||x - y||$. Then the open balls of radius $d/2$ centered at x and y respectively are disjoint open nbhds of x and y as you can check via the triangle inequality.

Example The cofinite topology on any infinite set X is not Hausdorff. Let $x, y \in X$. Then any nbhd of X excludes at most finitely many points, as is true for any neighborhood of y. As X is infinite, these two sets cannot be disjoint.

Example The discrete topology is Hausdorff as $\{x\}$ and $\{y\}$ are the suitable open sets.

Example If τ and τ' are two topologies on X, such that $\tau \subset \tau'$ (i.e. τ is finer than τ') and (X, τ) is Hausdorff, then (X, τ') is also Hausdorff.

Example The preceding implies that $\mathbb{R}_\ell$ is Hausdorff.

Example The preceding implies that $\mathbb{R}_K$ is Hausdorff.

Example If X is ordered then the order topology is Hausdorff.

Theorem 6. *Let X be a Hausdorff space. If $F \subset X$ is finite, then F is closed.*

Proof. We show that any singleton set in a Hausdorff space is closed. Then the fact that finite unions of closed sets are closed implies that finite sets are closed as well.

Let $x \in X$, and suppose that $y \in X$ is distinct from x. Select a neighborhood U of y which does not contain x (pick the neighborhood of x first if you like but it is superfluous). Thus y is not in the closure of $\{x\}$, and $\{x\} = \overline{\{x\}}$. So singleton sets are closed.

□

Many of the spaces we deal with are Hausdorff spaces, but a few are not. For example the cofinite topology on any infinite set is not Hausdorff. Yet in the cofinite topology sets of the form $\{x\}$ are closed, being the complement of open sets. So this nice behavior warrants a label of its own.

Definition If X is a space, and if $\{x\}$ is a closed set in X for every $x \in X$, then we say that X satisfies the T_1 axiom, and say that X is a T_1 space.

Example All Hausdorff spaces are T_1 spaces. The cofinite topology is a good example of a non-Hausdorff T_1 space.

Theorem 7. *Let X be a T_1 space, let $A \subset X$. Then $x \in X$ is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. We prove the contrapositive – if $x \in X$ has a nbhd U such that U contains only finitely many points of A, then x is not a limit point of A.

Suppose that such a U is given and let $a_1, \ldots, a_n$ be the points of $A \cap U$ which are distinct from x (x may or may not lie in A – it doesn't matter for the proof). Then the set $F = \{a_1, \ldots, a_n\}$ is a closed subset of X, and $U = X \setminus F$ is an open nbhd of x which satisfies $U \cap A \subset \{x\}$. But then U can not intersect A in a point different from x, and so x is not a limit point of A.

If every nbhd about x contains infinitely many points of A it certainly contains a point of A different from x, so that x is a limit point of A.

□

We'll talk more about the T_1 and other so-called "separation axioms" later in the course. They are mostly a register of pathology.

Theorem 8. *Let X be a Hausdorff space, let $(x_n) \subset X$ be a sequence. Then (x_n) converges to at most one point.*

Proof. Let $x_n \to y$ and $x_n \to x$. We prove that x and y are equal. For if they are not the same point, we can select nbhds U and V of x and y resp. such that $U \cap V = \emptyset$. If we pick N such that $n > N$ implies $x_n \in U$ and $x_n \in V$, we immediately obtain a contradiction. Thus $x = y$, and the limit of a sequence is unique (note that this does not imply the existence of a solution, only the uniqueness). □