MATH 54 - LECTURE

DAN CRYTSER

Introduction

In this lecture we examine the notion of convergence in a topological space. We see that in a topological spaces a sequence can converge to more than one point, and we introduce a condition, called the *Hausdorff condition*, that ensures this pathological behavior does not occur. In the second part of the lecture we review the notion of continuity for function $f : \mathbb{R}^n \to \mathbb{R}^m$ in terms of open sets. This leads us to the definition of continuity for function $f : X \to Y$ between topological spaces. We examine some of the properties of continuous functions.

Sequences

Definition If $(x_n) = x_1, x_2, \ldots$ is a sequence in a Euclidean space $\mathbb{R}^n$, we say that (x_n) converges to $x \in \mathbb{R}^n$ if for every $\epsilon > 0$, we can produce an integer $N > 0$ such that if $n > N$, then $||x_n - x|| < \epsilon$. That is, eventually the sequence enters and stays within any open ball about the point x.

The generalization of this to topological spaces replaces open balls with open sets.

Definition Let $(x_n) = x_1, x_2, \ldots$ be a sequence of elements $x_n \in X$, where X is a topological space. Then we say that (x_n) converges to the point $x \in X$ if for every nbhd U of x, we can produce an integer $N > 0$ such that $n > N$ implies that $x_n \in U$. That is, eventually the sequence enters and stays within any nbhd of x.

Theorem 1. *Suppose that X is a topological space with topology generated by the basis $\mathcal{B}$. Then a sequence (x_n) of points in X converges to $x \in X$ if and only if for every $B \in \mathcal{B}$ such that $x \in B$, there exists an integer $N > 0$ such that $n > N$ implies that $x_n \in B$.*

Example Let $X = \mathbb{R}$. The sequence $x_n = 1/n$ converges to 0 in the standard topology, because any basis element $(-\epsilon, \epsilon)$ contains all $1/n$ such that $1/n < \epsilon$.

Example Let $X = \mathbb{R}$. The sequence $x_n = 1/n$ does not converge to 0 in the K-topology, because $(-1, 1) - K$ is a nbhd of 0 which does not contain any points of (x_n).

Example Let $X = \mathbb{R}$. The sequence $x_n = 1/n$ does not converge to 0 in the upper limit topology, because $(-1, 0]$ is a basis element containing 0 which does not contain any points of the sequence (x_n).

Example Let $X = \mathbb{R} \times \mathbb{R}$ with the order topology coming from the dictionary topology. Then the sequence $a_n = 1/n \times 1/n$ does not converge to 0×0 in the order topology, because $(0 \times -1, 0 \times 1)$ (the vertical open interval containing $(0, 0)$) is a basis element in the order topology which contain 0×0 and none of the points in the sequence a_n.

Example The if $\tau_1 \subset \tau_2$ are two topologies on X, and $(x_n) \to x$ in (X, τ_2), what can we say about convergence in (X, τ_1)? What about the reverse situation?

Example Let $X = \{a, b\}$ and let $\tau = \{X, \emptyset, \{a\}\}$. Then τ is a topology. The sequence $a_n = a$ converges to a, as every point of this sequence belongs to $\{a\}$. However (a_n) also converges to b, for the only open set of X which contains b is $U = X$, and all points of the sequence (a_n) belong to this set.

Example Let X be any set and let τ be the concrete topology which has X and $\emptyset$ as its open sets. Then *every* sequence converges to *every* point.

Hausdorff Spaces

For many reasons we prefer sequences not to converge to multiple points. The following condition on a space (X, τ) forbids such behavior, and we will see several other such conditions in the future.

Definition Let X be a topological space. We say that X is a *Hausdorff space* or that X is *Hausdorff* if for every pair $x, y \in X$ such that $x \neq y$, there exists nbhds U and V of x and y respectively such that $U \cap V = \emptyset$.

Example Any Euclidean space $\mathbb{R}^n$ is a Hausdorff space, as if $x, y \in \mathbb{R}^n$, the open balls of radius $\epsilon = \frac{||x-y||}{2}$ about x and y give the required neighborhoods.

Example Any discrete space X is a Hausdorff space as the singletons $\{x\}$ and $\{y\}$ are the required nbhds.

Example The cofinite topology on $\mathbb{R}$ is not Hausdorff, as any non-empty open sets intersect in infinitely many points.

Example If $\tau_1 \subset \tau_2$ and (X, τ_1) is Hausdorff, then (X, τ_2) is Hausdorff. Thus $\mathbb{R}_K$ and $\mathbb{R}_\ell$ and $\mathbb{R}_U$ the upper limit topology are all Hausdorff, as they contain the standard topology on $\mathbb{R}$.

Example An exercise is to prove that any order topology is Hausdorff.

Now we can say why the Hausdorff condition forbids non-unique limits of sequences.

Theorem 2. *Let X be a Hausdorf space and (x_n) a sequence in X. Then (x_n) converges to at most one point of X.*

Proof. Suppose that $x_n \to x$ and $x \neq y$. We show that $x_n \not\to y$. Let U and V be disjoint nbhds of x and y. We can select an integer N such that $n > N$ implies that $x_n \in U$, because $x_n \to x$. Then $x_n \notin V$ for all $n > N$, so that the sequence (x_n) cannot converge to y. □

The Hausdorff condition is mild but many topological spaces fail to possess it. However, there is another simple condition which forbids some bad behavior.

Definition Let X be a space. We say that X satisfies the T_1 axiom if for every $x \in X$, the one-point set $\{x\}$ is a closed subset of X. A space that satisfies the T_1 axiom is called a T_1 space.

Example If we equip $\mathbb{R}$ with the cofinite topology the resulting space is not Hausdorff, but it is a T_1 space. In fact the cofinite topology is the smallest T_1 topology on any set, as you can check.

We aren't interested in this axiom right now, but we will see a large list of these axioms later in the course.

Continuous Functions

Recall that the principal theorem we proved about continuous functions was

Theorem 3. *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function between Euclidean spaces. Then the following are equivalent:*

(1) f is continuous at every point of $\mathbb{R}^n$.
(2) If $U \subset \mathbb{R}^m$ is open, then $f^{-1}(U) \subset \mathbb{R}^n$ is open.

As the definition of continuity of a function between Euclidean spaces requires the existence of the norm function $||\cdot||$, which is not defined everywhere, we can't formulate continuity for maps of topological spaces in terms of ϵs and δs.

Instead, we use the second formulation to give the following definition: **Definition** Let $f : X \to Y$ be a map between topological spaces. Then f is continuous if given any open set $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open in X.

Example Let X be a topological space. Then if $i_X : X \to X$ is the identity function, $i_X^{-1}(U) = U$ for any open set $U \subset X$. Thus the identity function is continuous.

Example Any constant function $f : X \to Y$ is continuous. Suppose that $f(X) = \{y\}$. Then $f^{-1}(U) = X$ if $y \in U$ and $f^{-1}(U) = \emptyset$ if $y \notin U$. Thus the preimage of an open set is open, and f is continuous.

Example Any function $f : \mathbb{R} \to \mathbb{R}$ which is continuous in the sense of calculus (i.e. $\epsilon - \delta$ continuous) is continuous in our sense, thanks to the theorem we proved in the first two lectures.

Example The identity function $i_X : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous, as the inverse image of $[0, 1)$ is $[0, 1)$ which is not open in $\mathbb{R}$ with the standard topology.

Theorem 4. *Let X and Y be top. spaces, let $f : X \to Y$. Then the following are equivalent:*
(a) f is continuous
(b) For every subset $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.
(c) For every closed set B of Y, $f^{-1}(B)$ is closed in X.
(d) For each $x \in X$, and each nbhd V of $f(x)$, there is a nbhd U of x such that $f(U) \subset V$.

Proof. (a implies b): Assume that f is continuous and that $y \in f(\overline{A})$. We show that $y \in \overline{f(A)}$. There exists $x \in \overline{A}$ such that $y = f(x)$. Let V be a nbhd containing $y = f(x)$. Then the pre-image $f^{-1}(V)$ is an open set containing x. As $x \in \overline{A}$, some point of A must be contained in $f^{-1}(V)$, say $a \in f^{-1}(V)$. Then $f(a) \in V$. Thus an arbitrary nbhd of y must contain an element of $f(a)$, and $y \in \overline{f(A)}$. Thus $f(\overline{A}) \subset \overline{f(A)}$.

(b implies c): Suppose that B is a closed subset of Y. Then $A = f^{-1}(B)$ is a subset of X, and by the assumption of (b) we obtain $f(\overline{A}) \subset \overline{f(A)}$. Suppose that $x \in \overline{A}$. Then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$. But $f(A) \subset B$, so we have $f(x) \subset \overline{f(A)} \subset \overline{B} = B$. Thus $x \in f^{-1}(B)$. So $\overline{A} \subset A$, and $A = f^{-1}(B)$ is closed.

(c implies d): Assume that preimages of closed sets are closed. Suppose that $x \in X$ and suppose that V is a nbhd of $f(x)$. We exhibit a nbhd U of x such that $f(U) \subset V$. The complement $Y \setminus V$ is closed, as V is open. Thus $f^{-1}(Y \setminus V)$ is closed in X. But the preimage of the complement is the complement of the preimage, i.e. $f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V)$. We can replace $f^{-1}(Y)$ with X, to obtain $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. So $X \setminus f^{-1}(V)$ is closed, so that $f^{-1}(V)$ is open. As $f(x) \in V$, we have $x \in f^{-1}(V)$. Thus $U = f^{-1}(V)$ is a nbhd of x such that $f(U) \subset V$.

(d implies a): Assume that such a U exists given such an x and V. We show that f is continuous. Suppose that $V \subset Y$ is an open set. If $f^{-1}(V)$ is empty, then it is open and we are done. So suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$, and we can produce U such that $f(U) \subset V$. We have shown that about any point of $f^{-1}(V)$ there exists an open set U such that $x \in U \subset f^{-1}(V)$. By the a previous HW problem this is the same as showing that $f^{-1}(V)$ is open.

$\square$

Homeomorphisms

Now that we have developed the notion of a continuous function, we can define what it means for two apparently distinct topological spaces to be "equal."

Recall the following definition:

Definition Let $f : X \to Y$ be a bijection between sets. The function $f^{-1} : Y \to X$ given by the rule $f^{-1}(y) = x$ if and only if $f(x) = y$ is called the inverse function or inverse of f.

Definition Let $f : X \to Y$ be a bijection between topological spaces. If f and f^{-1} are continuous, then f is called a homeomorphism, and X and Y are said to be homeomorphic if there exists a homeomorphism between X and Y.

Example Suppose that $f : X \to Y$ is a homeomorphism between top. spaces. Let g be the inverse of f. Then $f(U) = g^{-1}(U)$, for $g(f(U)) = i_X(U) = U$.

Example We show that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic. Let $F : (-1, 1) \to \mathbb{R}$ be defined by $F(x) = \frac{x}{1-x^2}$. Then F is a bijection with inverse $G(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$ as you can check. Both functions are continuous as they are built out of algebraic operations on $\mathbb{R}$, which are continuous (we will prove this rigourously later).

Example The identity function $id : \mathbb{R}_\ell \to \mathbb{R}$, where $\mathbb{R}$ has the std topology is continuous, for all open sets in $\mathbb{R}$ are open in $\mathbb{R}_\ell$, yet the inverse $id : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous, as $[0, 1)$ is not open in $\mathbb{R}$. Thus $id : \mathbb{R}_\ell \to \mathbb{R}$ is a continuous bijection without continuous inverse.