# MATH 54 - LECTURE

DAN CRYTSER

## Introduction

In this lecture we introduce the notion of a continuous function $f : X \to Y$, a generalization of the idea of a continuous function $g : \mathbb{R}^n \to \mathbb{R}^m$ between Euclidean spaces.

## Continuous Functions

Recall that the principal theorem we proved about continuous functions was

**Theorem 1.** *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a function between Euclidean spaces. Then the following are equivalent:*

*(1) $f$ is continuous at every point of $\mathbb{R}^n$.*
*(2) If $U \subset \mathbb{R}^m$ is open, then $f^{-1}(U) \subset \mathbb{R}^n$ is open.*

As the definition of continuity of a function between Euclidean spaces requires the existence of the norm function $|| \cdot ||$, which is not defined everywhere, we can't formulate continuity for maps of topological spaces in terms of $\epsilon$s and $\delta$s.

Instead, we use the second formulation to give the following definition: **Definition** Let $f : X \to Y$ be a map between topological spaces. Then $f$ is continuous if given any open set $U \subset Y$, the preimage $f^{-1}(U) \subset X$ is open in $X$.

**Example** Let $X$ be a topological space. Then if $i_X : X \to X$ is the identity function, $i_X^{-1}(U) = U$ for any open set $U \subset X$. Thus the identity function is continuous.

**Example** Any constant function $f : X \to Y$ is continuous. Suppose that $f(X) = \{y\}$. Then $f^{-1}(U) = X$ if $y \in U$ and $f^{-1}(U) = \emptyset$ if $y \notin U$. Thus the preimage of an open set is open, and $f$ is continuous.

**Example** Any function $f : \mathbb{R} \to \mathbb{R}$ which is continuous in the sense of calculus (i.e. $\epsilon - \delta$ continuous) is continuous in our sense, thanks to the theorem we proved in the first two lectures.

**Example** The identity function $i_X : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous, as the inverse image of $[0, 1)$ is $[0, 1)$ which is not open in $\mathbb{R}$ with the standard topology.

**Theorem 2.** *Let $X$ and $Y$ be top. spaces, let $f : X \to Y$. Then the following are equivalent:*
*(a) $f$ is continuous*
*(b) For every subset $A \subset X$, we have $f(\overline{A}) \subset \overline{f(A)}$.*
*(c) For every closed set $B$ of $Y$, $f^{-1}(B)$ is closed in $X$.*
*(d) For each $x \in X$, and each nbhd $V$ of $f(x)$, there is a nbhd $U$ of $x$ such that $f(U) \subset V$.*

*Proof.* (a implies b): Assume that $f$ is continuous and that $y \in f(\overline{A})$. We show that $y \in \overline{f(A)}$. There exists $x \in \overline{A}$ such that $y = f(x)$. Let $V$ be a nbhd containing $y = f(x)$. Then the pre-image $f^{-1}(V)$ is an open set containing $x$. As $x \in \overline{A}$, some point of $A$ must be contained in $f^{-1}(V)$, say $a \in f^{-1}(V)$. Then $f(a) \in V$. Thus an arbitrary nbhd of $y$ must contain an element of $f(a)$, and $y \in \overline{f(A)}$. Thus $f(\overline{A}) \subset \overline{f(A)}$.

(b implies c): Suppose that $B$ is a closed subset of $Y$. Then $A = f^{-1}(B)$ is a subset of $X$, and by the assumption of (b) we obtain $f(\overline{A}) \subset \overline{f(A)}$. Suppose that $x \in \overline{A}$. Then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$. But $f(A) \subset B$, so we have $f(x) \subset \overline{f(A)} \subset \overline{B} = B$. Thus $x \in f^{-1}(B)$. So $\overline{A} \subset A$, and $A = f^{-1}(B)$ is closed.

(c implies d): Assume that preimages of closed sets are closed. Suppose that $x \in X$ and suppose that $V$ is a nbhd of $f(x)$. We exhibit a nbhd $U$ of $x$ such that $f(U) \subset V$. The complement $Y \setminus V$ is closed, as $V$ is open. Thus $f^{-1}(Y \setminus V)$ is closed in $X$. But the preimage of the complement is the complement of the preimage, i.e. $f^{-1}(Y \setminus V) = f^{-1}(Y) \setminus f^{-1}(V)$. We can replace $f^{-1}(Y)$ with $X$, to obtain $f^{-1}(Y \setminus V) = X \setminus f^{-1}(V)$. So $X \setminus f^{-1}(V)$ is closed, so that $f^{-1}(V)$ is open. As $f(x) \in V$, we have $x \in f^{-1}(V)$. Thus $U = f^{-1}(V)$ is a nbhd of $x$ such that $f(U) \subset V$.

(d implies a): Assume that such a $U$ exists given such an $x$ and $V$. We show that $f$ is continuous. Suppose that $V \subset Y$ is an open set. If $f^{-1}(V)$ is empty, then it is open and we are done. So suppose that $x \in f^{-1}(V)$. Then $f(x) \in V$, and we can produce $U$ such that $f(U) \subset V$. We have shown that about any point of $f^{-1}(V)$ there exists an open set $U$ such that $x \in U \subset f^{-1}(V)$. By the a previous HW problem this is the same as showing that $f^{-1}(V)$ is open.

□

## More Properties

There are several properties which hold for any continuous function between any pair of spaces.

**Theorem 3.** *Let $X, Y, Z$ be spaces.*

*(1) Any constant function $f : X \to Y$ is continuous.*
*(2) If $A \subset X$, and we equip $A$ with the subspace topology, then the inclusion $j : A \to X$ is continuous (in fact, this is the largest topology on $A$ making this map continuous).*
*(3) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*
*(4) The restriction $f|_A$ of a continuous function $f : X \to Y$ to any subspace $A \subset X$ is also continuous ($A$ is given the subspace topology).*
*(5) Suppose that $f : X \to Y$ is continuous, and $f(X) \subset W \subset Y \subset Z$ as subspaces (all the topologies are subspace topologies). Then we can consider $f : X \to W$ or $f : X \to Z$, and both of these maps are continuous. In particular, $f : X \to f(X)$ is a continuous surjection.*
*(6) If $f : X \to Y$ is a function, and $X = \cup_i U_i$ is a union of open subsets, such that $f|_{U_i} : U_i \to Y$ is continuous for each $U_i$, then $f$ is continuous.*

## Pasting Lemma

If we have two continuous functions defined on subspaces of $X$, such that the union of their domains covers $X$ and such that the restriction to the intersection of the domains agrees, we can paste them together to obtain a continuous function on the whole space as follows.

**Theorem 4.** *Let $X = A \cup B$, where $A$ and $B$ are closed subspaces of $X$. If there are continuous functions $f : A \to Y$ and $g : B \to Y$ such that $f|_{A\cap B} = g|_{A\cap B}$, then we can define $h : X \to Y$ by $h|_A = f$ and $h|_B = g$. The function $h$ is continuous.*

*Proof.* The function is well defined by the intersection property. Suppose that $C \subset Y$ is closed. We show that $h^{-1}(C)$ is closed also. But $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$, a finite union of closed sets, as the functions $f$ and $g$ are continuous. □

## Homeomorphisms

Now that we have developed the notion of a continuous function, we can define what it means for two apparently distinct topological spaces to be "equal."

Recall the following definition:

**Definition** Let $f : X \to Y$ be a bijection between sets. The function $f^{-1} : Y \to X$ given by the rule $f^{-1}(y) = x$ if and only if $f(x) = y$ is called the inverse function or inverse of $f$.

**Definition** Let $f : X \to Y$ be a bijection between topological spaces. If $f$ and $f^{-1}$ are continuous, then $f$ is called a homeomorphism, and $X$ and $Y$ are said to be homeomorphic if there exists a homeomorphism between $X$ and $Y$.

**Example** Suppose that $f : X \to Y$ is a homeomorphism between top. spaces. Let $g$ be the inverse of $f$. Then $f(U) = g^{-1}(U)$, for $g(f(U)) = i_X(U) = U$.

**Example** We show that $(-1, 1)$ and $\mathbb{R}$ are homeomorphic. Let $F : (-1, 1) \to \mathbb{R}$ be defined by $F(x) = \frac{x}{1-x^2}$. Then $F$ is a bijection with inverse $G(y) = \frac{2y}{1+(1+4y^2)^{1/2}}$ as you can check. Both functions are continuous as they are built out of algebraic operations on $\mathbb{R}$, which are continuous (we will prove this rigourously later).

**Example** The identity function $id : \mathbb{R}_\ell \to \mathbb{R}$, where $\mathbb{R}$ has the std topology is continuous, for all open sets in $\mathbb{R}$ are open in $\mathbb{R}_\ell$, yet the inverse $id : \mathbb{R} \to \mathbb{R}_\ell$ is not continuous, as $[0, 1)$ is not open in $\mathbb{R}$. Thus $id : \mathbb{R}_\ell \to \mathbb{R}$ is a continuous bijection without continuous inverse.

## Maps into Products

If you have a product topological space $X \times Y$, and maps $f : A \to X$ and $g : A \to Y$, you can encode all of the information stored in the maps $f$ and $g$ in one big map from $A$ to $X \times Y$, as follows:

**Theorem 5.** *Suppose that $A, X, Y$ are topological spaces, and $f_1 : A \to X$ and $f_2 : A \to Y$ are continuous maps. Then there exists a unique map $f : A \to X \times Y$ such that $\pi_i \circ f = f_i$. Conversely, if $f : A \to X \times Y$ is continuous, then the maps $f_i = \pi_i \circ f : A \to X_i$ are continuous.*

To prove this we need a lemma:

**Lemma 1.** *The canonical projection maps $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are continuous.*

*Proof.* If $U \subset X$ is an open subset of $X$, then $\pi_1^{-1}(U) = U \times Y$ is a basis element in the product topology, hence open. The same proof works for $\pi_2$. □

*Proof.* (of Theorem): Suppose that $f_1, f_2$ are given. Notice that the requirement that $\pi_i \circ f = f_i$ forces us to define $f(a) = (f_1(a), f_2(a))$. We show that the preimage of a basis element under $f$ is open in $A$, i.e. if $U \subset X$ and $V \subset Y$ are open, then $f^{-1}(U \times V)$ is also open.

**Claim**: $f^{-1}(U \times V) = f_1^{-1}(U) \cap f^{-1}(V)$ (these are both subsets of $A$). For if $f(a) \in U \times V$, then the $x$-coordinate of $f(a)$ belongs to $U$, so that $f_1(a) = \pi_1 \circ f(a) \in U$. So $a \in f_1^{-1}(U)$. Similarly, $a \in f_2^{-1}(V)$. Thus $f^{-1}(U \times V) \subset f_1^{-1}(U) \cap f^{-1}(V)$. Showing the other direction is similar. So the claim is proved. Now we have written the preimage of $U \cap V$ as the intersection of preimages $f_1^{-1}(U) \cap f_2^{-1}(V)$. As $f_1$ and $f_2$ are themselves continuous, these preimages are both open subsets of $A$. The finite intersection of open sets is open, hence $f^{-1}(U \times V)$ is open. As the preimage of basis element in $X \times Y$ is open, $f$ is continuous.

If we are given $f : A \to X \times Y$, the fact that the functions $f_i = \pi_i \circ f$ are continuous just amounts to the continuity of a composition of continuous functions.

$\square$