# MATH 54 - HW SOLUTIONS 3

DAN CRYTSER

## Introduction

In this lecture we talk about embeddings and introduce the product topology on the arbitrary product $\prod_i X_i$ of a collection of topological spaces. We also define the box topology on such a topology, and show that the box topology is finer than the product topology in the case when the index set is infinite. We exhibit the first instance of a "universal mapping property" – a sort of function from diagrams to diagrams.

## Embeddings

Let $f : X \to Y$ be an injective continuous function between topological spaces. Then $f$ is also a continuous bijective function $f : X \to f(X)$, where you replace the codomain with the range. **Definition** We say that a continuous injective function $f : X \to Y$ is an embedding if it the bijective continuous function $f : X \to f(X)$ has continuous inverse.

**Example** The map that sends $\mathbb{R}$ to the $y$-axis in $\mathbb{R}^2$ is not a homeomorphism, but it is a homeomorphism onto the range.

**Example** The map $f : [0, 1) \to \mathbb{R}^2$ which sends $x$ to $e^{2\pi ix}$ is continuous, as it is composed out of algebraic and transcendental functions. It's also injective, as you can check. The range of $f$ is the unit circle $\mathbb{T} = \{x \in \mathbb{R}^2 : ||x|| = 1\}$ in $\mathbb{R}^2$. The inverse map $f^{-1} : \mathbb{T} \to [0, 1)$ is NOT continuous, as the pre-image of the open set $[0, 1/2)$ is not open in the circle (the circle inherits the Euclidean topology from $\mathbb{R}^2$). Later we will see that there can be NO homeomorphism between $[0, 1)$ and $\mathbb{T}$.

## Arbitrary Products of Sets

**Definition** Let $\{X_i\}_{i\in J}$ be a family of (non-empty) sets. The product $\prod_i X_i$ is defined to be the collection of all function $s : J \to \cup_i X_i$ such that $s(i) \in X_i$ for all $i \in J$. An element $s \in \prod_{i\in J} X_i$ is often called a $J$-tuple.

This generalizes the notion of product as ordered pair or list or sequence of elements – the index set $J$ is not required to be countable.

**Example** Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then $f \in \prod_{i\in\mathbb{R}} \mathbb{R}$.

Our goal in this section is to investigate the different ways in which the product $\prod_i X_i$ may be endowed with a topology when all the sets $X_i$ are topological spaces.

## The Product Topology

First we generalize the canonical projection maps from the finite product case.

**Definition** Let $\{X_i\}_{i\in J}$ be a collection of sets indexed by some set $J$. If $j \in J$, then the $j$th projection map $\pi_j : \prod_i X_i \to X_j$ is given by $\pi_j(s) = s(j)$.

**Example** In the example above, if $x \in \mathbb{R}$, then $\pi_x(f) = f(x)$, so that the projection map just "evaluates at $x$."

Note that the sets $\mathcal{S} = \{\pi_j^{-1}(U_j) : j \in J, U_j \subset X_j \text{ is open }\}$ cover the product $\prod_i X_i$, as $s \in \prod_i X_i$ is also in $\pi_j^{-1}(X_j)$ for any $j \in J$.

**Definition** Let $\{X_j\}_{j \in J}$ be a collection of topological spaces. We define the *product topology* on $\prod_i X_i$ to be the topology generated by the subbasis $\mathcal{S}$ above.

We can formulate this in terms of a basis as follows:

**Definition** Let $\{X_j\}_{j \in J}$ be a collection of topological spaces. We define the *product topology* on $\prod_i X_i$ to be the topology generated by the basis $\mathcal{B} = \{\prod_i U_i : U_i \subset X_i \text{is open and } U_i = X_i \text{ for all but finitely many } i\}$.

**Definition** A subset $U \subset \prod_i X_i$ is open in the product topology if and only if about any point $x \in U$, there exists $\{U_i : U_i \subset X_i\}$ such that $x_i \in U_i$, $U_i = X_i$ for all but finitely many $i \in I$, and $\prod U_i \subset U$.

**Example** Let $J = \mathbb{N}$ and let $X_n = \mathbb{R}$ for all $n$. Then $\prod_n \mathbb{R}$ is usually denoted $\mathbb{R}^\omega$ ($\omega$ often denotes an infinite ordered or "ordinal" object in set theory). The set $U = \{f \in \mathbb{R}^\omega : f(1) > 0\}$ is open in the product topology on $\mathbb{R}^\omega$, as it is the basis element $\prod_n U_n$ which has $U_1 = (0, \infty)$ and all other $U_n = \mathbb{R}$.

**Example** The set $U$ of all $f \in \mathbb{R}^\omega$ such that $f(n) > 0$ for all $n \in \mathbb{N}$ is NOT open, as there is no basis element $U_1 \times U_2 \ldots U_m \times \mathbb{R} \times \mathbb{R} \times \ldots$ containined within $U$ (such a basis element would contain sequences with negative entries).

**Example** The product topology on $\mathbb{R} \times \mathbb{R}$ (finite product) is the same as the finite product topology that we discussed, as you can check. This is the same as the standard topology on $\mathbb{R}^2$.

## The Box Topology

You likely have wondered why we didn't just take the basis to be $\mathcal{B}' = \{\prod U_i | U_i \subset X_i \text{ is open }\}$, as this is simpler to define than the basis we used for the product topology. The answer is that $\mathcal{B}'$ yields *too many* open sets on the product set $\prod_i X_i$.

**Definition** The topology defined on $\prod_i X_i$ by the basis above is called the *box topology* on $\prod_i X_i$.

The following example might shed some light on why we dislike this space:

**Example** Let $f : \mathbb{R} \to \prod_{n=1}^{\infty} \mathbb{R}$ be given by $f(t) = (t, t, t, t, \ldots)$. We equip $\mathbb{R}$ with the standard topology and $\prod_{n=1}^{\infty} \mathbb{R}$ with the box topology. Then $f$ is not continuous! For the set $V = \prod_{n=1}^{\infty}(-1/n, 1/n)$ is an open set in the codomain which contains $f(0) = (0, 0, 0, \ldots)$. Yet there is no nbhd $U = (-\epsilon, \epsilon)$ of 0 such that $f(U) \subset V$, because this would force $\epsilon < 1/n$ for all $n$, a contradiction. Thus $f$ is not continuous.

If we give $\prod_{n=1}^{\infty} \mathbb{R}$ the product topology, then $U = \prod_{n=1}^{\infty}(-1/n, 1/n)$ is not an open set, as there is no basis element containing 0 and contained within $U$. For a basis element in the product topology looks like $U_1 \times U_2 \times \ldots \times U_k \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \ldots$, where the finite list $U_1, \ldots, U_k$ is allowed to have proper open subsets of $\mathbb{R}$, but all the factors after that must be all of $\mathbb{R}$.

In fact, we can see that the mapping is continuous when the codomain is given the product topology via the following theorem, which tells us exactly when a map into a product is continuous (there is good reason to believe that there is no such neat criterion which works in all cases for determining when a map out of a product space is continuous).

## The Universal Mapping Property

First we note that each of the projection mappings $\pi_i : \prod_j X_j \to X_i$ is continuous when $\prod_i X_i$ is given the product topology (also if the product is given the box topology, but this is not relevant).

**Lemma 1.** *Let $X = \prod_j X_j$, and let $\pi_i : X \to X_i$ be the canonical projection onto the ith factor, i.e. $\pi_i(f) = f(i)$. If $X$ is equipped with the product topology, then $\pi_i$ is continuous.*

*Proof.* Suppose that $U_i \subset X_i$ is continuous. We show that $\pi_i^{-1}(U_i)$ is open in the product topology. But $\pi_i^{-1}(U_i)$ is simply the basis element in the product topology $\prod V_j$ which has $V_i = U_i$ and all other $V_j = X_j$. This is open as it is a basis element in the product topology. □

The product topology is a tool for converting a collection of continuous maps into one ”large” continuous map which holds all of their information. We call it a ”universal mapping property” because the recipe which produces this large map does not vary from collection to collection; we are always following the same algorithm to produce the large map, and it is always unique. This is made precise in the following theorem, which we state a bit differently from Munkres (owing to that worldliness and sophistication which we possess, as a class):

**Theorem 1.** *Let $A$ be a space, let $\{X_i\}_{i \in I}$ be a family of spaces, and suppose that we are given, for each $i \in I$, a continuous mapping $f_i : A \to X_i$. Then there is a unique continuous map $f : A \to \prod_i X_i$ such that $\pi_i \circ f = f_i$ for all $i \in I$. Conversely, if $f : A \to \prod_i X_i$ is continuous, then each of the maps $f_i = \pi_i \circ f$ is continuous.*

*Proof.* The requirement that $\pi_i \circ f = f_i$ forces $f(a)(i) = f_i(a)$ (recall that $f(a) \in \prod_i X_i$ and so $f(a)$ is a function defined on the $i$). Thus if the function exists, it is unique. We show that $f$ which carries to $a$ to $f(a)$ is continuous. Let $U = \prod_i U_i$ be a basis element of $\prod_i X_i$ in the product topology, so that $U_i = X_i$ for all but finitely many $i$. Then $f^{-1}(U) = \cap_i f_i^{-1}(U_i)$, as you can check. This is an open set. You might say “Dan, stop this. You are attempting to claim that the arbitrary intersection of open sets is open, and even a fourth-grader knows this to be false.” But I know that all but finitely many of the $U_i$ are equal to their total spaces $X_i$, so that $f_i^{-1}(U_i) = A$ for all but finitely many $i$, and so the intersection is really taken over a finite collection of open subsets of $A$ (intersecting finitely many subsets of $A$ with a bunch of copies of $A$ does not change anything – the intersection is the same). Thus $f^{-1}(U)$ is open in $A$, and so $f$ is continuous.

If the function $f$ is taken to exist, then the functions $f_i = \pi_i \circ f$ are all continuous because they are compositions of continuous functions. □