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A Piano's Dilemma: Loudness vs. Harmonicity

A tremendous dilemma involved in a piano's acoustical engineering is how to preserve the harmonicity of its tone while having enough loudness to be heard by its listeners in a room of considerable size. To understand the solution to this problem, one must first grasp the concept of wave impedance and the beneficial application of beating.

When a disturbance, or wave, is initiated in a medium and travels toward the boundary between this medium and another medium, such as when a wave travels along a slender wire to a thicker wire or a soundboard, a certain fraction of the wave is transmitted into the new medium and the remainder is reflected back into the original medium. The amplitudes of the reflected and the transmitted waves, and the amount of energy carried by these now two waves, all depend on the ratio of the wave impedances of the two media (French, 1971). If the new medium's impedance is significantly larger than that of the original medium, then there is almost complete reflection with only a small portion of the total energy sent on to the new medium. However, if the two media have wave impedances that are approximately equal, then there is very little reflection and the disturbance is almost completely transmitted across the junction. Wave impedance depends on the same two properties of the medium that wave velocity does, but in the wave impedance formula, these two properties are arranged differently.

$$c_{string} = \sqrt{\frac{T}{\frac{m}{L}}} = \sqrt{\frac{T}{(\pi r^2 d)}} \quad , \quad d = \quad material \quad density$$

$$Z_{string} = \sqrt{(\pi r^2 d)T} \quad , \quad Z = \quad wave \quad impedance$$

One can trade tension for density or radius while keeping the impedance the same, but it is impossible at the same time to keep the traveling speed, and thus also the pitch, unchanged. Beneath the strings is the soundboard, whose wave impedance is much larger than that of a string. At its driving point, the soundboard's wave impedance can be expressed by the following formula:

$$Z_{soundboard} = t^2 \sqrt{Y_w d_w} \times (a),$$

a = a numerical constant

t = thickness of the soundboard

d_w = density of the wood

Y_w = modulus of elasticity of the wood

The wave impedance ratio between the struck string and the soundboard must be chosen to meet two conflicting requirements. On the one hand, there must be sufficient transmission of vibratory energy from the string to the soundboard so the human ear can hear a sufficiently loud sound. On the other hand, the soundboard wave impedance has to be high enough that its resonances will not play an "unacceptably large role" in tuning of individual string modes (Benade, 1990). After optimizing the soundboard's wave impedance, one should consider the wave impedance of the strings, as the transmission between the two media depends on the ratio of the two wave impedances.

$$\frac{Z_{string}}{Z_{soundboard}} = \left(\frac{r^2}{t^2}\right) \times a \, ,$$

a = a numerical constant

r = string radius

t = thickness of the soundboard

As one considers the radius of the string in this matter, one should also keep in mind the inharmonicity involved. An ideal string vibrates in a series of modes that are harmonics of a fundamental. In actuality, strings have some stiffness, which provides a restoring force, in addition to the tension, slightly raising the frequency of all the modes. The additional restoring force is greater in the case of the higher modes because the string makes more wiggles. Thus the modes are spread apart in frequency and are no longer exact harmonics of a fundamental. In short, a real string with stiffness is partly ideal-string-like and partly rigid-rod-like (Rossing, 2002). The inharmonicity of strings, the amount by which the actual mode frequencies differ from a harmonic series, is found to vary with the square of the partial number (Fletcher, 1964).

$$f_n = nf_1[1 + (n^2 - 1)A],$$

f_n = frequency of the nth harmonic

f_1 = fundamental frequency

A = string stiffness

For a solid wire without wrapping,

$$A = \frac{\pi^3 r^4 E}{8TL^2},$$

A = string stiffness

r = string radius

E = Young's modulus

T = tension

L = string length

Therefore, the inharmonicity is the smallest for long, thin wires under great tension. Since the stiffness of the string increases sharply with its radius ($A \propto r^4$), one should use the thinnest possible string, as L has already been fixed by the frequency requirements laid down for the string. However, if the string were too thin, the piano sound would be barely audible, for the transmission of vibration from the string to the soundboard is proportional to the wave impedance ratio, which depends on the wire radius to soundboard thickness.

$$\left(\frac{r^2}{t^2}\right) \times a = \frac{Z_{string}}{Z_{soundboard}},$$

a = a numerical constant

r = string radius

t = thickness of the soundboard

A solution to the problem of simultaneously striving for harmonicity, which calls for thin strings, and loudness is to use several strings, each with acceptable inharmonicity and the ability to join with the others in driving the soundboard to a greater vibrational amplitude. However, it turns out that the employment of multistring coupling further elicits several considerations for decay time versus harmonicity.

Fine-tuning of the coupled strings of the unisons is a means of regulating the amount of decaying partials and decay time. Studies have shown that the best piano sound results from tuning coupled strings several cents different from each other (Kirk, 1959). If the strings are tuned to exactly the same frequency, the transfer of energy from the strings to the soundboard takes place rapidly, and the decay time of the sound is too small. If the unison strings are tuned too far apart, prominent beats are heard, resulting the internal physiology of a barroom piano. Roger Kirk of the Baldwin Piano Company finds that piano tuners and musicians are unanimous in their verdict that too-close tuning gives a tone that not only sounds dead and lacks warmth but also dies away too rapidly. Laboratory measurements have since confirmed the subjective auditory impressions that slightly detuned strings, which are considered “normal,” die away in about the same total length of time as a single string in the coupled set when the others are prevented from vibrating. However, when three strings are tuned exactly together they die away much more rapidly. The presence of other precisely in tune strings encourages each string to transfer its vibration more rapidly to the soundboard and hence the room.

Using the impedance formulas, one can deduce the decreasing decay time (wave impedance ∝ decay time) by a factor of the number of strings that are coupled together. Three identically tuned strings would stay precisely in step with one another, and there would be no frictional or other force acting between them to change either their physical condition or movement in case they did touch. These three closely spaced strings would behave exactly the same when they were far apart, and when their sides were fused together in a unified planar tricord formation, and in a way that the total cross-sectional area stays constant. The aggregate impedances would be the same (Benade, 1990). The total combined tension acting on the composite tricord is three times the tension acting on each of the original strings, and the mass is tripled as well because there are now three strings instead of one.

$$Z_{tricord} = \sqrt{3(\pi r^2 d) \times 3T} = 3 \times \left(Z_{one \quad string} \right)$$

Therefore, for a coupled set of three strings, the wave impedance is increased by a factor of 3, which diminishes the decay time in a proportional manner.

After surveying the preferences of a large group of people, Kirk found that while both 2-cent and 8-cent deviation tuning methods are acceptable, trained musicians prefer less deviation than do the untrained. The beat frequencies between the first five partials of two C4 strings tuned 2 cents and 8 cents apart is as follows.

Difference at *n*-th harmonic	1	2	3	4	5
2-cent difference (in Hz)	0.3	0.61	0.91	1.2	1.50
8-cent difference (in Hz)	1.21	2.42	3.53	4.8	6.10

One can see that with the 2-cent detuning the beating rate for the first pair of partials is quite slow, as are those for the second and third pair of partials. As a result, the tone sounds reasonably smooth when played by itself. The 8-cent spread gives a rather brighter sound but is not yet the sort of jangle one gets with a spread of 15 to 20 cents (Benade, 1990).

Nevertheless, there is a trade-off of musical virtues between the two kinds of tuning methods as one compares various musical intervals. When using a 2-cent detuning between strings, the 0.91-Hz beat frequency belonging to its set of the third partials is just able to cover up the 0.89-Hz beat that one uses in setting the equal-temperament fifth to G4. The G4 strings' second partial would have a similar beating rate to further obscure the departure from deviation-free tuning. With the 8-cent detuning method, on the other hand, the fifths become quite diffuse. Using the 2-cent detuning method, in the case of an equal temperament major third interval, the fifth partials of C4 strings have within them a 1.5-Hz maximum beating frequency, as do the fourth partials of E4, if E4's strings also have a 2-cent detuning spread. One can see that there is the possibility of beat frequencies as high as 1.5 Hz + 1.5 Hz = 3 Hz among the partials upon which an interval can be based. The beating rate for a piano tuner's third in equal temperament is about 8 Hz, a little more than 2 times of 3Hz. If the spread among members of a three-string "unison" were increased to 8 cents, the beating would be come rapid enough to drown the temperament error completely (Benade, 1990).

As seen above, both wave impedance and beating play a critical role in retaining as much harmonicity as possible while allowing enough resonance and beating to generate and to maintain enough loudness for a listener to hear the tone produced by the piano. While many still fastidiously seek to improve its design, the modern piano, is filled with the marvels of mechanical and acoustical engineering and technology.

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