Expectation and Variance: Continuous Random Variables

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Expected Value

**Definition.** Let $X$ be a real-valued random variable with density function $f(x)$. The expected value $\mu = E(X)$ is defined by

$$\mu = E(X) = \int_{-\infty}^{+\infty} x f(x) \, dx ,$$

provided the integral

$$\int_{-\infty}^{+\infty} |x| f(x) \, dx$$

is finite.
Properties

• If $X$ and $Y$ are real-valued random variables and $c$ is any constant, then

$$E(X + Y) = E(X) + E(Y),$$

$$E(cX) = cE(X).$$

• More generally, if $X_1$, $X_2$, ..., $X_n$ are $n$ real-valued random variables, and $c_1, c_2, ..., c_n$ are $n$ constants, then

$$E(c_1X_1 + c_2X_2 + \cdots + c_nX_n) = c_1E(X_1) + c_2E(X_2) + \cdots + c_nE(X_n).$$
Example

• Suppose Mr. and Mrs. Lockhorn agree to meet at the Hanover Inn between 5:00 and 6:00 P.M. on Tuesday.

• Suppose each arrives at a time between 5:00 and 6:00 chosen at random with uniform probability.

• Let $Z$ be the random variable which describes the length of time that the first to arrive has to wait for the other.

• What is $E(Z)$?
Expectation of a Function of a Random Variable

**Theorem.** If $X$ is a real-valued random variable and if $\phi : \mathbb{R} \to \mathbb{R}$ is a continuous real-valued function, then

$$E(\phi(X)) = \int_{-\infty}^{+\infty} \phi(x) f_X(x) \, dx,$$

provided the integral exists.
Expectation of the Product of Two Random Variables

**Theorem.** Let $X$ and $Y$ be independent real-valued continuous random variables with finite expected values. Then we have

$$E(XY) = E(X)E(Y).$$
Example

• Let $Z = (X, Y)$ be a point chosen at random in the unit square.

• What is $E(X^2Y^2)$?
Variance

Definition. Let $X$ be a real-valued random variable with density function $f(x)$. The variance $\sigma^2 = V(X)$ is defined by

$$\sigma^2 = V(X) = E((X - \mu)^2).$$
Computation

**Theorem.** If $X$ is a real-valued random variable with $E(X) = \mu$, then

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx.$$
Properties of the variance

- If $X$ is a real-valued random variable defined on $\Omega$ and $c$ is any constant, then

\[
V(cX) = c^2V(X),
\]
\[
V(X + c) = V(X).
\]
• If $X$ is a real-valued random variable with $E(X) = \mu$, then

$$V(X) = E(X^2) - \mu^2.$$
• If $X$ and $Y$ are independent real-valued random variables on $\Omega$, then

$$V(X + Y) = V(X) + V(Y).$$
Example

- Let $X$ be an exponentially distributed random variable with parameter $\lambda$.

- Then the density function of $X$ is

$$f_X(x) = \lambda e^{-\lambda x}.$$ 

- What is $E(X)$ and $V(X)$?
Normal Density

• Let $Z$ be a standard normal random variable with density function

$$ f_Z(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. $$

• What us $E(X)$ and $V(X)$?
Cauchy Density

- Let $X$ be a continuous random variable with the Cauchy density function
  \[ f_X(x) = \frac{a}{\pi a^2 + x^2} \cdot \]
- What is $E(X)$ and $V(X)$?
Independent Trials

**Theorem.** If $X_1, X_2, \ldots, X_n$ is an independent trials process of real-valued random variables, with $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, and if

$$S_n = X_1 + X_2 + \cdots + X_n,$$
$$A_n = \frac{S_n}{n},$$

then

$$E(S_n) = n\mu,$$
$$E(A_n) = \mu,$$
$$V(S_n) = n\sigma^2,$$
\[ V(A_n) = \frac{\sigma^2}{n} . \]

It follows that if we set

\[ S_n^* = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} , \]

then

\[ E(S_n^*) = 0 , \]
\[ V(S_n^*) = 1 . \]

We say that \( S_n^* \) is a standardized version of \( S_n \).