Math 68. Algebraic Combinatorics. Fall 2005

Topics for the final project (version of 10/14/05)

Each student should work on a project throughout the quarter. It is strongly encouraged that you work together in groups of two or three students. At the end of the quarter, each group will present their project giving a short talk. Here is a list of possible topics. If you have your own topic in mind you are more than welcome to discuss it with me.

1. A *descent* of a permutation $\pi \in \mathcal{S}_n$ is an index i $(1 \leq i \leq n-1)$ such that $\pi_i > \pi_{i+1}$. The *major index* of π, denoted $\mathrm{maj}(\pi)$, is defined as the sum of all descents in π. For example, $\mathrm{maj}(12\cdots n) = 0$ and $\mathrm{maj}(n\cdots 21) = 1+2+\cdots+(n-1)$.
   Show that the major index maj is equidistributed with the number of inversions inv, that is,
   $$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{maj}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{inv}(\pi)}.$$
   A stronger statement that you may also try to prove is that the joint distribution is symmetric, that is,
   $$\sum_{\pi \in \mathcal{S}_n} q^{\mathrm{maj}(\pi)} t^{\mathrm{inv}(\pi)} = \sum_{\pi \in \mathcal{S}_n} q^{\mathrm{inv}(\pi)} t^{\mathrm{maj}(\pi)}.$$
   - D. Foata and M.-P. Schutzenberger, Major index and inversion number of permutations, *Math. Nach.* 83 (1978), 143–159.
2. **The Gessel-Viennot method.** This is a beautiful formula to enumerate n-tuples of nonintersecting lattice paths. The answer is given by a determinant, and the proof is based on the combinatorics of involutions.
   - [EC1] Section 2.7.
   - I. Gessel and G. Viennot, Binomial determinants, paths, and hook length formulae, *Advances in Math.* 58 (1985), 300–321.
3. **Combinatorial proofs of the Lagrange Inversion Formula.** In class we will see an analytic proof of the Lagrange Inversion Formula. However, it is possible to give a combinatorial proof, interpreting the coefficients as counting plane forests.
   - [EC2] Section 5.4.
4. **Walks in graphs.** The number of walks of given length between two vertices of a graph can be expressed in terms of the eigenvalues of its adjacency matrix.
   - [St] Section 1.
5. **The transfer-matrix method.** An application of counting walks in graphs to other problems in enumerative combinatorics.
   - [EC1] Section 4.7.
6. **Viennot's geometric construction of the RSK correspondence.** In class we will discuss the RSK algorithm, which gives a correspondence between permutations and

pairs of tableaux. A beautiful geometric description of this correspondence is due to Viennot.

- Section 3.6 of [Bruce E. Sagan, *The Symmetric group*, Springer, second edition, 2001].

7. **Increasing and decreasing subsequences of permutations.** This theory is an application of the Robinson-Schensted correspondence (or RSK algorithm).
   - [BS] Section 5.

8. Read a paper from a combinatorics journal and present it in class. Here are some interesting journals that you can find in the library or online.
   - *Journal of Combinatorial Theory A*,
   - *Electronic Journal of Combinatorics*,
   - *Journal of Algebraic Combinatorics*,
   - *European Journal of Combinatorics*,
   - *Annals of Combinatorics*,
   - *Discrete Mathematics*.

9. **Open problem.** Let $f(n,k)$ be the number of ways to draw $k(n-2k-1)$ diagonals (the maximum possible) between vertices of a convex n-gon P so that (a) the vertices of each edge are at distance at least k+1 apart, and (b) there do not exist $k+1$ edges intersecting pairwise in their interiors. Then $f(n,k)$ is the number of non-crossing k-tuples of Dyck paths from $(0,0)$ to $(2n-4k,0)$. This fact has been proved in [J. Jonsson, Generalized triangulations and diagonal-free subsets of half-moon shapes, preprint, `http://www.math.kth.se/~jakobj/combin.html#deltank`], but no bijective proof is known. Any progress on this problem would be interesting.