EXERCISES FOR FINAL EXAM

These exercises constitute the take-home final exam in Math 68.

- Solutions are due by the end of the day on Friday, December 11th.
- But, please note that I will be out of town from Wednesday, December 9th until Friday, December 11th and therefore unable to offer any assistance.
- You may consult reference books and the interwebs, but please do not copy answers from these sources.
- This exam is untimed.
- Your solutions to this exam will be graded on a finer scale than regular homework problems, more like your solutions to the midterm exam were graded.
- Therefore, please make sure that the steps in your solutions are explained.

1. Find an explicit formula for $a_n$ if
   \[ a_{n+2} = 5a_{n+1} - 6a_n + 2n - 1 \]
   for $n \geq 0$ with initial conditions $a_0 = 0$ and $a_1 = 1$. 
   Note: here and in other problems, you are allowed to use a computer algebra program, providing that you also turn in its output.

2. Count unlabeled graphs on 5 vertices by their number of edges using Pólya’s Theorem (like we did in Lecture 25). Include a chart describing the action of $S_5$ on edges.

3. Calculate the number of spanning trees of the $6 \times 6$ torus graph $G$. The vertices of this graph are ordered pairs in $\mathbb{Z}_6 \times \mathbb{Z}_6$, where $(i, j)$ is connected to $(i, j - 1), (i, j + 1), (i - 1, j), (i + 1, j)$, mod 6.

4. Find an explicit formula for the number of trees on $n$ labeled nodes with exactly 4 leaves. Hint: The answer involves Stirling numbers, although there may be other ways to do the problem. Second hint: You may want to investigate Prüfer codes.

5. Consider expanding the product
   \[ \prod_{1 \leq i < j \leq n} (x_i + x_j) \]
   into monomials. Prove that the number of monomials in this expansion equals the number of forests on $n$ labelled vertices. For example there are 2 forests on 2 labeled vertices, which correspond to the monomials $x_1$ and $x_2$, while there are 7 forests on 3 labeled vertices, corresponding to the 7 terms of the expansion of $(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)$:
   \[ x_1^2x_2 + x_1x_2^2 + x_1x_3 + x_1x_3^2 + x_2x_3 + x_2x_3^2 + x_3^2 + 2x_1x_2x_3. \]