Möbius Inversion (and Inclusion-Exclusion)

Ex 7.1: There are:
- 45 students
- 14 play (S)occer
- 17 play (B)asketball
- 4 play S & B.

How many play neither?

Solution:

A bit of formalism.

Define $g: 2^{S, B, H} \to \mathbb{N}$
by $g(X) = \# \text{students who play all sports in } X$ (but possibly more)

Also define $f: 2^{S, B, H} \to \mathbb{N}$
by $f(X) = \# \text{students who play precisely the sports in } X$.

Note: $g(X) = \sum_{Y \subseteq X} f(Y)$.

Inclusion-Exclusion: $f(\emptyset) = \sum_{Y \subseteq \emptyset} (-1)^{|Y|} g(Y)$.

Ex 7.2

45 students
- 14 play (S)occer,
- 17 play (B)asketball,
- 19 play (H)ockey,
- 4 play S & B,
- 3 play S & H,
- 5 play B & H,
- 1 plays S, B, & H.

So we are "inverting" a function which maps from $2^{S, B, H}$ to $\mathbb{N}$.

But why not generalize?

Consider functions from any poset to any ring.
In number theory, if 
\[ g(n) = \sum \frac{f(d)}{d|n} \]
then 
\[ \phi(n) = \sum \mu\left(\frac{n}{d}\right) g(d) \]
where 
\[ \mu\left(\frac{n}{d}\right) = \begin{cases} (-1)^s & \text{if } \frac{n}{d} \text{ is the product of } \ell \text{ distinct primes,} \\ 0 & \text{otherwise.} \end{cases} \]

\( \mu \) is called the Möbius function.

Instead of \( \mu\left(\frac{n}{d}\right) \), we'll consider \( \mu(d, n) \).

Really, this is just linear algebra.

Returning to Example 7.2, let's consider matrices & vectors indexed by subsets of \( \{5, 8, 11, 13\} \). We need to pick a standard order. Let's choose

\[ \emptyset, 5, 8, 11, 13, 58, 511, 811, 5811. \]

Then...

More generally, for any poset \( P \), the incidence algebra \( I(P) \) is the set of all matrices \( M \) indexed by elements of \( P \) such that \( M(x, y) = 0 \) unless \( x \leq y \).

We want to invert

\[ \zeta(x, y) = \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases} \]

The inverse, \( \mu(x, y) \), is called the Möbius function of \( P \).

While we could use row operation (or something else) to invert, there is a better way.
\[ m \zeta = \text{Id} \]

Look at its \((x, y)\) entry:

\[
\sum_{z \in I(P)} m(x, z) \zeta(z, y)
\]

\[ = \sum_{z \neq y} m(x, z) \]

Would like \(m\) in \(I(P)\), so...

\[ = \sum_{x \leq y} m(x, z) \]

\[ = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \]

This shows that we can define \(m\) inductively.

\[
m(x, x) = 1
\]

\[
m(x, y) = 0 \text{ if } x \nleq y
\]

Otherwise, \(x < y\), so

\[
0 = \sum_{x \leq y} m(x, z)
\]

\[ = \sum_{x \leq y} m(x, z) + m(x, y), \text{ for } x \leq y \]

so we can recursively define

\[
m(x, y) = -\sum_{x \leq y} m(x, z).
\]

**The Principle of Möbius Inversion**

Suppose \(f\) and \(g\) are functions from the poset \(P\) to any ring and satisfy

\[ g(x) = \sum_{y \leq x} f(y) \]

for all \(x \in P\). Then

\[ f(x) = \sum_{y \leq x} m(x, y) g(y). \]

**Proof:** \(g = \zeta f\), so \(f = \mu g\). \(\Box\)

**Ex:**

\[ P = \begin{array}{c}
\text{c} \\
\text{b} \\
\text{d} \\
\text{a}
\end{array} \]

Fix an ordering on the vertices of \(P\). Preferably, this should be a linear extension, \(a, b, c, d, e\).

Then

\[
m = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

Now fill in the rest.