Last lecture:

The Principle of Möbius Inversion

Let \( P \) be a locally finite poset, and \( f, g : P \to R \) (some ring).

If

\[
g(x) = \sum_{y \in P} f(y)
\]

for all \( x \in P \), then

\[
f(x) = \sum_{y \in P} \mu(x, y) g(y),
\]

where \( \mu \) is the Möbius function of \( P \).

Other direction:

If

\[
g(x) = \sum_{y \in P} f(y)
\]

for all \( x \in P \), then

\[
f(x) = \sum_{y \in P} \mu(y, x) g(y).
\]

Ex: \( P = (N, \leq) \). Then...

\[
\begin{array}{cccc}
& a & b & c & d & e \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{array}
\]

\( \mu \):

\[
\sum_{x \leq y} \mu(x, z) = 0.
\]

This last equation shows how to define \( \mu \) recursively, by

\[
\mu(x, y) = -\sum_{x \leq y} \mu(x, y).
\]

Proof: By definition, \( \mu(x, x) = 1 \)
and \( \mu(x, y) = 0 \) if \( x \not= y \), so it suffices to check that

\[
\sum_{x \leq y} \mu(x, z) = 0
\]

if \( x < y \). But this is clear, since this sum reduces to \( \mu(x, z) \cdot \mu(y, z) \).
Note: If \( g(n) = \sum_{i \leq n} f(i) \), then
\[
 f(n) = \sum_{i \leq n} \mu(i,n) g(i)
 = -g(n-1) + g(n)
 = g(n) - g(n-1).
\]

**Ex 16.19** Let \( P = (a^n, S) \). Then:
\[
\mu(S,T) = \begin{cases} 
(-1)^{|T-S|} & \text{if } S \subseteq T \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof:** We must verify that this formula satisfies
\[
\mu(S, S) = 1 \quad \text{(clearly true)},
\mu(S, T) = 0 \quad \text{if } S \not\subseteq T \quad \text{(clearly true), and}
\sum_{S \subseteq T} \mu(S, T) = 0 \quad \text{if } S \subseteq T.
\]

This last equation reduces to
\[
\sum_{i=0}^{|T-S|} (-1)^i \binom{|T-S|}{i} = 0,
\]
which follows from the Binomial Theorem. \(\blacksquare\)

**Ex 16.20** Let \( P \) be the set of positive integers under the divisor order.
Then:
\[
\mu(x, y) = \begin{cases} 
(-1)^r & \text{if } \frac{y}{x} \text{ is the product of } r \text{ distinct primes,} \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof:** Clearly \( \mu(x, x) = 1 \) because \( \frac{y}{x} \) is the product of 0 distinct primes. Also, \( \mu(x, y) = 0 \) if \( x \nmid y \). Therefore it suffices to check that
\[
\sum_{x \mid y} \mu(x, z) = 0
\]
when \( x \mid y \).

But we have \( \mu(x, z) = 0 \) whenever a square divides \( \frac{y}{x} \), so
\[
\sum_{x \mid y} \mu(x, z) = \sum_{x \mid y} \mu(x, z)
= \sum_{x \mid z} \mu(x, z)
\]
when \( \frac{y}{x} \) is square-free.

Now let \( p_1, \ldots, p_k \) denote the set of distinct primes that divide \( \frac{y}{x} \).
There are \( \binom{k}{i} \) integers \( z \) such that \( x \mid z \) and \( \frac{y}{x} \) is the product of \( i \) distinct primes, so this sum reduces to
\[
\sum_{x \mid y} \mu(x, z) = \sum_{i=0}^k \binom{k}{i} (-1)^i \binom{i}{i-1},
\]
which is 0 by the Binomial Theorem. \(\blacksquare\)
Interlude: A million dollar question.

The Mertens function is
\[ M(n) = \sum_{1 \leq i \leq n} \mu(1, i). \]

If for every \( \varepsilon > 0 \) there is a constant \( C \) so that
\[ M(n) < C n^{1/2 + \varepsilon} \]
then the Riemann hypothesis is true.

A common generalization of these two:
submultisets of a multiset.

Let \( M \) be a multiset. Then the Mobi\'us function for all submultisets of \( M \) is
\[ \mu(S, T) = \begin{cases} (-1)^{|T-S|} & \text{if } S \subseteq T \text{ and } T \setminus S \text{ has no repeated elements} \\ 0 & \text{otherwise} \end{cases} \]

The proof is similar to the divisor poset proof.

Products of Posets
Let \( P_1 \) and \( P_2 \) be two posets.
Their product, \( P_1 \times P_2 \), is the poset defined on ordered pairs \( (x_1, x_2) \) with \( x_1 \in P_1 \) and \( x_2 \in P_2 \) in which \( (x_1, x_2) \leq (y_1, y_2) \) if and only if \( x_1 \leq y_1 \) (in \( P_1 \)) and \( x_2 \leq y_2 \) (in \( P_2 \)).

Isomorphism
Two posets \( P_1 \) and \( P_2 \) are isomorphic if there is a bijection
\[ \psi: P_1 \to P_2 \]
such that \( x \leq y \) in \( P_1 \) if and only if \( \psi(x) \leq \psi(y) \) in \( P_2 \).

Claim: Our previous examples are isomorphic to products.

Ex: Divisors of 72: \( 2^3 \cdot 3^2 \) ordered by divisibility:

\[ \begin{array}{c}
72 = 2^3 \cdot 3^2 \\
2^3 \cdot 3^2 \\
2^3 \cdot 3^2 \\
2 \end{array} \]

\[ \begin{array}{c}
(1, 1) \\
(3, 1) \\
(3, 1) \\
(1, 1) \\
(1, 1) \\
(3, 0) \\
(1, 0) \\
(1, 0) \\
(0, 0) \\
(0, 0) \\
(0, 0) \\
(10, 1, 1) \end{array} \]
Example: Subsets of \{1,2,3\} ordered by \subseteq:

\[\begin{array}{c}
1 & 12 & 13 & 23 & 123 \\
& 1 & 12 & 13 & 23 \\
& & 1 & 12 & 13 \\
& & & 1 & 12 \\
\end{array}\]

And:

\[\begin{array}{c}
1 & (1,0) & (1,0) & (0,1) & (0,1) \\
& 1 & (1,0) & (0,1) & (0,1) \\
& & 1 & (0,1) & (0,1) \\
& & & 1 & (0,1) \\
\end{array}\]

\((10,1), \subseteq^3\)

So, it would be nice to figure out the Möbius functions of products...

Product Formula 16.24

Let \(P = P_1 \times P_2\). The Möbius function of \(P\) is given by:

\[\mu_P((x_1,x_2),(y_1,y_2)) = \mu_{P_1}(x_1,y_1) \mu_{P_2}(x_2,y_2).\]

Proof: Clearly this function satisfies

\[\mu_P((x_1,x_2),(x_1,x_2)) = 1\]

and \(\mu_P((x_1,x_2),(y_1,y_2)) = 0\) if \((x_1,x_2) \not\leq (y_1,y_2)\). So it suffices to prove that for \((x_1,x_2)\leq (y_1,y_2)\),

\[\sum_{(x_1,x_2)\leq (y_1,y_2)} \mu_P((x_1,x_2),(y_1,y_2)) \mu_{P_1}(x_1,y_1) \mu_{P_2}(x_2,y_2) = 0\]

But this sum can be rewritten:

\[\left(\sum_{x_1 \leq y_1} \mu(x_1,y_1)\right) \left(\sum_{x_2 \leq y_2} \mu(x_2,y_2)\right)\]

so since either \(x_1 < y_1\) or \(x_2 < y_2\) (or both), we are done. \(\blacksquare\)