Antichains in \((2^{[n]}, \leq)\)

Sperner's Theorem (1928): If \(A \subseteq 2^{[n]}\) is an antichain, then \(|A| \leq \binom{n}{\lfloor n/2 \rfloor}\).

Proof: We use the "LYM technique" of Lubell, Yamamoto, and Meshalkin from \(\sim 1954\).

A chain is maximal if no other element can be added to it without destroying the chain property. The poset \((2^{[n]}, \leq)\) clearly has \(n!\) maximal chains

\[ \phi = A_0 \subset A_1 \subset A_2 \subset \ldots \subset A_n = [n]. \]

Also, if \(A\) is a set of size \(k\), then \(k! (n-k)!\) of these maximal chains include \(A\). Finally, if \(A \subseteq 2^{[n]}\) is an antichain, then every saturated chain can include at most one element of \(A\), so:

\[ n! = \# \text{ maximal chains} \]
\[ \geq \# \text{ maximal chains which include a member of } A, \]
\[ = \sum_{A \subseteq A} |A|! (n-|A|)! \]
\[ = n! \sum_{A \subseteq A} \frac{1}{(|A|)!}. \]

This expression is minimized when \((|A|) = \binom{n}{\lfloor n/2 \rfloor}\), so we see that

\[ |A| / \binom{n}{\lfloor n/2 \rfloor} \leq 1. \]

An application of Sperner's Theorem

Littlewood-Offord (1943): Suppose \(z_1, z_2, \ldots, z_n \in C\) satisfy \(|z_k| \geq 1\) for every \(k\). Then the number of sums of the form \(\sum_{k \in [n]} E_k z_k\) where \(E_k \in \{-1, 1\}\) which lie inside the unit circle is at most

\[ C \frac{2^n \log n}{\sqrt{n}}, \]

for some constant \(C > 0\).

Erdős (1945): Suppose \(x_1, x_2, \ldots, x_n \in \mathbb{R}\) satisfy \(|x_k| \leq 1\) for every \(k\). Then the number of sums of the form \(\sum_{k \in [n]} E_k x_k\) where \(E_k \in \{-1, 1\}\) which lie inside any open interval of length \(\delta\) at most \(\left(\frac{n^2}{\sqrt{n}}\right)^{\frac{C n}{2}}\).

Proof: Fix an open interval \(I \subseteq \mathbb{R}\) of length \(\delta\). WLOG, we may assume that each \(x_k\) is positive. For any \(A \subseteq [n]\), we define

\[ S(A) = \sum_{k \in A} x_k - \sum_{k \notin A} x_k. \]

Note that the \(S(A)\) quantities are precisely the sums we are interested in.
If \( A \subseteq B \) then \( S(B) > S(A) \); in fact, \( S(B) \geq S(A) + 2 \). Therefore at most one can lie in \( I \), so the collection
\[
\{ A \subseteq [n] : S(A) \in I \}
\]
forms an antichain. The result now follows from Sperner's Theorem. \( \Box \)

To get the full Littlewood-Offord result, we will need to generalize Sperner's Theorem. While the LYM technique gave a beautiful proof of Sperner's Theorem, we need to use a more combinatorial proof for the generalization. As an added bonus, this combinatorial proof also shows that the binomial coefficients are unimodal.

### Symmetric Chain Decompositions

We begin with an example, for \( 2^{[4]} \):

![Symmetric Chain Decomposition Diagram]

A chain \( A_1 \subseteq A_2 \subseteq \ldots \subseteq A_k \subseteq [n] \) is a symmetric chain if
1. \( |A_{k+1} \setminus A_k| = 1 \) for all \( k \), and
2. \( |A_1| + |A_k| = n \).

A symmetric chain decomposition (SCD) of \( (2^{[n]}, \subseteq) \) is a collection of disjoint symmetric chains whose union is \( (2^{[n]}, \subseteq) \).

It had been known that \( (2^{[n]}, \subseteq) \) has an SCD since the early 1950's. The most explicit, and frequently used, SCD for \( (2^{[n]}, \subseteq) \) is due to Greene and Kleitman from 1976.

To specify an SCD, we need to describe:
- how to go "up" if we don't stop
- how to go "up" if we don't stop
- how to go "up" if we don't stop
- how to go "up" if we don't stop

We do this by specifying a successor function
\[
\sigma : 2^{[n]} \to 2^{[n]} \cup \{ \text{stop} \}.\]
The set \( A \in \text{[n]} \) has characteristic vector \( \chi(A) \) in which the \( k^{th} \) component is 1 if \( k \in A \) and 0 otherwise.

Given \( \chi(A) \), we match the 0's and 1's from left to right: when a 0 is encountered, it becomes an unmatched 0; when a 1 is encountered, it gets matched to the rightmost unmatched 0 (which becomes matched).

**Ex:** \( A = \{3, 4, 6, 7, 8\} \subseteq \text{[9]} \)
\[ \chi(A) = \overbrace{0 0 0 1 0 0 1 1 0}^{\text{unmatched 0}} \]

Now we can describe the successor function:
\[ \sigma(A) = \begin{cases} 
\text{stop} & \text{if no unmatched 0s} \\
A \cup \{k\} & \text{if \( k \) is the leftmost unmatched 0} 
\end{cases} \]

**Ex:** \( \sigma(\{3, 4, 6, 7, 8\}) = \{3, 4, 6, 7, 8, 9\} \).
\[ \chi(\{3, 4, 6, 7, 8, 9\}) = \overbrace{0 0 1 1 0 1 1 1}^{\text{unmatched 0}} \]
This has no unmatched 0s, so the chain stops here.

**Greene-Kleitman (1976):** This construction gives an SCD for \( (2^{\text{[n]}}, \subseteq) \).

**Proof:** Homework.

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We now strengthen Sperner’s Theorem:

**Katona and Kleitman (independent, 1965):**
Choose any partition \( X/Y \) of \( [n] \) into nonempty parts, and a collection \( A \subseteq 2^{[n]} \). If there are no indices \( j \neq k \) so that both
- \( A_j \subset A_k \) and
- \( A_k \setminus A_j \subseteq X \text{ or } Y \),
then \( |A| \leq \binom{\lceil \frac{n}{2} \rceil}{\lfloor \frac{n}{2} \rfloor} \).

**Proof:** Consider SCDs for both \( X \) and \( Y \). For any pair of chains
- \( C: E_1 \subseteq E_2 \subseteq \ldots \subseteq E_g \) and
- \( D: F_1 \subseteq F_2 \subseteq \ldots \subseteq F_h \),
we form the “symmetric rectangle”
\[ E_1 \cup F_1 \quad E_1 \cup F_2 \quad \ldots \quad E_1 \cup F_h \]
\[ E_g \cup F_1 \quad \ldots \quad E_g \cup F_h \].

If \( A \) satisfies the hypotheses, then we can have at most one member of \( A \) in any row or column, so the number of members of \( A \) which occur above is bounded by \( \min(g, h) \).
By homework problem 7, \( \min(g,h) \) is precisely the number of subsets of size \( n/2 \) which occur in this rectangle.

Therefore, by considering all such symmetric rectangles, we see that

\[
|a| \leq \left( \frac{n}{2} \right). \]

Finally, we return to the Littlewood-Offord problem.

**Katona and Kleitman (independent, 1965):**

Suppose \( z_1, z_2, \ldots, z_n \in \mathbb{C} \) satisfy \( |z_k|^2 \leq 1 \) for every \( k \). Then the number of sums of the form \( \sum_{k \in [n]} \varepsilon_k z_k \) where \( \varepsilon_k \in \{\pm 1\} \) which lie inside the unit circle is at most \( \left( \frac{n}{2} \right) \).

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**Proof:** WLOG, we may assume that \( \Re z_k \) is nonnegative for every \( k \).

Set

\[
X = \{ k : \Im z_k \geq 0 \}, \quad Y = \{ k : \Im z_k < 0 \}.
\]

For \( A \subseteq [n] \), we define

\[
S(A) = \sum_{k \in A} z_k - \sum_{k \not\in A} z_k.
\]

If \( A \not\subseteq B \) and \( B \setminus A \subseteq X \), then \( S(A) \) and \( S(B) \) differ by a set of complex numbers all in the first quadrant and all of norm at least 2, so at most one of \( S(A) \) and \( S(B) \) can lie inside the unit circle.

The same holds if \( A \subseteq B \) and \( B \setminus A \subseteq Y \), so the collection

\[
A = \{ A \subseteq [n] : |S(A)| < 1 \}
\]

satisfies the hypotheses of the previous theorem, showing that

\[
|a| \leq \left( \frac{n}{2} \right). \]