Bootstrap Percolation

Begin with a matrix $M$ of 0s and 1s.

At each stage, if two or more neighbors of an entry are 1s, that entry becomes a 1.

This process continues indefinitely.

(Do example)

(Look at MathWorld)

Typical percolation question: if the initial matrix is random, with probability of a 1 = $p$, what happens?

(Note: Prof. Winkler teaches Math 100 in winter term — percolation.)

Shapiro & Stevens (1991):

What if the initial matrix is a permutation matrix?

When does $M_r$ fill up?

Ex: $r = 24378165$ does not fill up.

Ex: $r = 16724358$ does fill up.

How many fill up?

Towards a Characterization...

An interval in the permutation $\pi$ is a set $I$ of contiguous indices such that $\pi(I) = \{\pi(i) : i \in I\}$ is also contiguous.

Ex: $24378165$

Ex: $16724358$

Every permutation $\pi \in S_n$ has $n$ intervals of length 1 and 1 interval of length $n$. If $\pi$ has no other intervals, then it is called simple.

Inflations

Given $\sigma \in S_m$ and nonempty permutations $\sigma_1, \ldots, \sigma_m$, the inflation

$\sigma[\sigma_1, \ldots, \sigma_m]$

is the permutation obtained by each entry $\sigma(i)$ by an interval in the same relative order as $\sigma_i$.

Ex: $24378165 = 2413[132, 12, 1, 2]$

Ex: $16724358 = 12[1672435, 1]$

$= 12[1, 5613247]$. 
**Uniqueness**

Every permutation except \( 1 \) is the inflation of a unique simple permutation of length at least 2.

**Proof:** Consider the maximal proper (i.e., not whole thing) intervals of a permutation.

If two of these intersect, then their union must be the whole permutation, In this case the permutation is the inflation of 12 or 21.

Otherwise, the maximal proper intervals are disjoint and, by maximality, define a simple permutation. ■

**Permutations that don't fill up**

**Observation:** If \( \sigma \) of length 24 is simple, then \( M\sigma \) does not fill up. In fact, bootstrap percolation leaves \( M\sigma \) unchanged.

**Proof:** Suppose that the entry in position \((ij)\) is changed from 0 to 1 in the first iteration of bootstrap percolation. Then:

\[
\begin{array}{c|c|c|c|c}
 & B & A \backslash & & D \\
\hline
A & 1 & 0 & & \\
\hline
\end{array}
\]

at least 2 of A, B, C, or D are filled in \( M\sigma \). If A is filled, then B or C must be filled. But this implies that \( \sigma \) isn't simple. ■

**A sufficient condition**

If \( \pi \) and \( \sigma \) both fill up, then 12[\( \pi, \sigma \)] and 21[\( \pi, \sigma \)] both fill up.

**Proof:**

\[
12[\pi, \sigma] = \begin{array}{c}
\pi \\
\downarrow \\
\end{array}
\]

**Extending this...**

**Observation:** If \( \sigma \) does not fill up, then for any choice of nonempty \( a_1, \ldots, a_m \), \( \sigma[a_1, \ldots, a_m] \) also does not fill up.

**Proof:** Consider

\[
\begin{array}{c|c|c|c|c}
 & a_1 & a_2 & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 & a_1 & a_2 & & \\
\hline
\end{array}
\]

\[
\begin{array}{c|c|c|c|c}
 & a_1 & a_2 & & \\
\hline
\end{array}
\]
Characterization

Theorem: The permutation \( \pi \) fills up under bootstrap percolation if and only if \( \pi \) can be built from the permutation \( 1 \) using the operations

\[ \sigma \oplus \tau = 12[\sigma, \tau] \quad \text{and} \quad \sigma \circ \tau = 21[\sigma, \tau]. \]

Definition: These permutations are called \textit{separable}.

How many are there?

Review from last time

132-avoiding permutations:

\[
\begin{align*}
\text{empty perm} & \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{empty_perm}}
\end{array} \\
\downarrow & \\
1 & + \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{132_avoiding}}
\end{array} \\
\downarrow & \\
\quad & \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{f_x_f}}
\end{array} \\
& \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{f}}
\end{array}
\end{align*}
\]

\[ f = \frac{1 - \sqrt{1 - 4x}}{2x}. \]

Counting

Separable permutations:

\[
\begin{array}{c}
\text{empty} \\
\downarrow \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{empty_perm}}
\end{array} \\
\end{array}
\]

Problem: the \textit{and} \textit{decompositions aren't unique.}

Solution: \( \pi \) is \textit{\( \Theta \)-decomposable} if it can't be written as \( \sigma \oplus \tau \) for nonempty \( \sigma \) and \( \tau \).

Analogous: \( \Theta \)-indecomposable.

Uniqueness

If \( \pi \) is \( \Theta \)-decomposable, then there is a unique \( \Theta \)-indecomposable permutation \( \sigma \) such that

\[ \pi = \sigma \oplus \tau. \]

Separable permutations:

\[
\begin{align*}
\{13\} & \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{13}}
\end{array} \\
\downarrow & \\
\quad & \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{g_f}}
\end{array} \\
& \quad \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{h_f}}
\end{array}
\end{align*}
\]

where

\[ f = g.f. \text{ for separables} \]
\[ g = g.f. \text{ for } \Theta \text{-ind. separables} \]
\[ h = g.f. \text{ for } \Theta \text{-ind. separables} \]

Note: none of these count empty perm.
Now note:

\[ g = \text{ind. separables} \]
\[ = \text{separables} - \quad \square \]
\[ = f - gf \]

So:

\[ g(1+f) = f, \]
\[ g = \frac{f}{1+f}. \]

Exactly the same:

\[ h = \frac{f}{1+f}. \]

Therefore:

\[ f = x + \frac{2f^2}{1+f}. \]

Solving

\[ f^2 + f = x + xf + 2f^2 \]
\[ 0 = f^2 + (x-1)f + x \]

\[ f = \frac{1-x \pm \sqrt{1 - 6x + x^2}}{2} \]

Which to choose?
Remember: we excluded the empty permutation, so \( f(0) \neq 0 \).

These are the large Schröder numbers.