Graphs

Formal definition: a graph is a set $V$ of vertices equipped with a set $E$ of size 2 subsets of $V$ called edges.

Ex: $V = \{1, 2, 3\}$
$E = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$

We write $i \sim j$ if $\{i, j\} \in E$.

Sometimes we...
- allow loops,
- direct the edges,
- allow multiple edges between two vertices,
- put weights on the edges.

But usually we study simple graphs.

An independent set in a graph is a subset $I \subseteq V$ of vertices, no two adjacent.

A clique is a subset $C \subseteq V$ of vertices, where there is a edge connecting every pair of vertices.

Ex: independent set: $\{2, 3\}$, $\{1, 3\}$
clique: $\{1, 2, 3\}$, $\{3, 4\}$.

Ramsey’s Theorem: Fix $k$. Every sufficiently large graph contains either a clique or an independent set with at least $k$ vertices.

An independent set is maximal (an MIS) if it is not properly contained in another independent set.

Ex: Let $C_n$ denote the cycle on the vertices $[n]$ where
$i \sim j \iff i = j \pm 1$ (mod $n$)

$C_5:
\begin{array}{ccc}
1 & 2 & 3 \\
\downarrow & \uparrow & \downarrow \\
4 & 5 & 1
\end{array}$

How many MISes does $C_n$ have?

Define $a_n = \#MISes$ in $C_n$, so:

$a_2 = 2$
$a_3 = 3$
$a_4 = 2$

Proposition: For $n \geq 5$, $a_n = a_{n-3} + a_{n-2}$.

Proof: Consider an MIS $I \subseteq C_n$.

Since $n \geq 5$, $I \subseteq Z$. Let $i < j$ denote the greatest two elements of $I$.

Clearly $j-i$ is either 2 or 3. If $j-i = 2$, then removing $j$ gives an MIS of $C_{n-2}$. If $j-i = 3$, then removing $j$ gives an MIS of $C_{n-3}$.

Inverting this map is easy. For every MIS $I \subseteq C_{n-2}$, let $i$ denote its greatest entry and add $i+2$. For $I \subseteq C_{n-3}$, add $i+3$. \qed
These are the Perrin numbers.

**Perrin's Theorem:** If $p$ is prime, then $p \mid a_p$.

**Proof:** Midterm.

The least non-prime $n$ such that $n \mid a_n$ was found in 1982; it is $521^2 = 271441$.

**Erdős in 1965:** How many MISes can a graph on $n$ vertices have?

Define

$$m(G) = \# \text{MISes in } G$$

$$g(n) = \max \{ m(G) : G \text{ has } n \text{ vertices} \}.$$

**Ex:**

$$G = \triangle, \quad m(G) = 3.$$

$$H = \mathcal{L}, \quad m(H) = 2.$$

$$G \cup H = \triangle \mathcal{L}, \quad m(G \cup H) = 6.$$

**Proposition:** For all graphs $G$ and $H$, we have

$$m(G \cup H) = m(G) \cdot m(H).$$

**Proof:** For any MIS $I \subseteq G \cup H$,

$I \cap G$ must be an MIS of $G$, and

$I \cap H$ must be an MIS of $H$.

Conversely, if $I$ is an MIS of $G$ and $J$ is an MIS of $H$, then $I \cup J$ is an MIS of $G \cup H$. □

**Conjecture:** For all $n \geq 2$,

$$g(n) = \begin{cases} 3^k & n = 3k \\ 4 \cdot 3^{k-1} & n = 3k + 1 \\ 2 \cdot 3^k & n = 3k + 2. \end{cases}$$

Achieved by disjoint unions of

\( \mathcal{L}, \quad \triangle, \quad \bigtriangleup \), and \( \bigtriangleup \)

**Moon-Moser:** This is correct.

First, we need a technical lemma.
For any vertex $v$ of $G$, its open neighborhood is
\[ N(v) = \{ u : u \sim v \} . \]
Its closed neighborhood is
\[ N[v] = \{ u \} \cup N(u) . \]

Proposition: For any vertex $v \in G$,
\[ m(G) \leq m(G-v) + m(G-N[v]) . \]

Proof: Take an MIS $I \subseteq G$. If $v \in I$, then $I \setminus v$ is an MIS of $G-N[v]$, and conversely, if $J$ is an MIS of $G-N[v]$, then $J \cup \{v\}$ is a MIS of $G$. Every MIS $I$ of $G$ which doesn't contain $v$ is also an MIS of $G-v$. \[ \square \]