1. In this problem, you may cite familiar facts from linear algebra without proving them. For instance, you do not need to prove that matrix multiplication is associative.
(a) Is the set $S = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) : a \in \mathbf{R}\}$ with matrix multiplication as the law of composition a group?
(b) Same question but with the set $S = \{\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right) : a \in \mathbf{R}^{\times}\}$.

2. Assume that G is a group with every $x \in G$ satisfying $x^2 = 1$, the identity element. Show that G is abelian.

3. Identify (as one of the subgroups we've considered) the kernel of the homomorphism $\varphi\colon G \to Aut(G)$ defined by $a \mapsto (g \mapsto aga^{-1})$.

4. Is $\mathcal{P}_n = \{$set of permutation matrices in $\mathrm{GL_n}(\mathbf{R})\}$. Is $\mathcal{P}_n$ a normal subgroup of $GL_n(\mathbf{R})$?

5. Assume $H \leq G$ and let $g \in G$.
(a) Prove that $gHg^{-1} = \{ghg^{-1} : h \in H\}$ is a subgroup of G of the same order as H.
(b) Show that if H is the only subgroup of G with order $|H|$, then H is a normal subgroup of G.
(Hint: Part (a) can be done quickly using problem 3, page 72. For part (b), you may use problem 13(b), page 72.)

6. What is $Z(S_3)$, the center of the symmetric group S_3?

7. Let T be the set of equilateral triangles in the plane with the equivalence relation $s \sim t \Leftrightarrow s$ is congruent to t . Define a function $T \to X$, where X is a familiar set, such that there is a bijection $T/\!\sim\ \longrightarrow X$ induced by f.

8. Let G be a group. The image of the homomorphism φ of problem 3 above is called the inner automorphism group of G, and denoted by $Inn(G)$. Prove that $G/Z(G) \approx Inn(G)$. To what familiar group is $Inn(S_3)$ isomorphic?

9. Let G be a group with normal subgroups H and K. Assume $K \subseteq H$. Then $H/K \leq G/K$. Prove that H/K is a normal subgroup of G/K and that $\frac{G/K}{H/K} \approx G/H$. (*Hint:* Define a homomorphism $G/K \to G/H$ whose kernel is H/K, and use the first isomorphism theorem.) To what familiar group is $\frac{\mathbf{Z}/60\mathbf{Z}}{\langle 15\rangle}$ isomorphic?

10. Show that $\mathbf{Q}^+$, the group of rationals under addition, is not the direct product of two nontrivial groups.

11. Identify the quotient group $H/Z(H)$, where H is the quaternion group and $Z(H)$ its center.

12. Homomorphisms of cyclic groups
    (a) Explicitly describe all homomorphisms from $\mathbf{Z}/n\mathbf{Z}$ to $\mathbf{Z}$, where n is a positive integer.
    (b) For which positive integers m and n is there a homomorphism $\mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ given by sending $1 + m\mathbf{Z}$ to $1 + n\mathbf{Z}$?