13. Let $C$ be a cyclic group, and let $\varphi: G \to C$ be any surjective homomorphism with $\ker \varphi \subseteq Z(G)$. Show that $G$ is abelian. Hint: Show every $g \in G$ can be written as $zx^i$ where $z \in Z(G)$ and $C = \langle \varphi(x) \rangle$.

14. Show that if a group $G$ has a subgroup of finite index, then $G$ has a normal subgroup of finite index. What is an upper bound for the index of the normal subgroup in terms of the index of the subgroup? Hint: Consider the action of $G$ on $G/H$, where $H \leq G$ has finite index.

15. Let $S$ be a $G$-set. If $s \in S$ and $a \in G$, show that the stabilizers of $s$ and $a \cdot s$ are related by $G_{as} = aG_s a^{-1}$.

16. If a group $G$ has order $pq$ where $p$ and $q$ are (not necessarily distinct) primes, show that either $G$ is abelian or $Z(G) = \{1\}$. (Hint: Use problem 13 above.) Can both possibilities occur?

17. What is the class equation of the dihedral group $D_5$?

18. Is $D_4 \cong H$, where $H$ is the quaternion group?

19. Find all subgroups of $\mathbb{Z}$ containing $12\mathbb{Z}$ and arrange them in a lattice. Then use the correspondence theorem to exhibit the lattice of all subgroups of $\mathbb{Z}/12\mathbb{Z}$.

20. Show that every group of order $p^3$, where $p$ is a prime, has normal subgroups of order $p$ and $p^2$.

21. Assume that $R$ and $R'$ are rings and $\varphi: R \to R'$ is a function which preserves addition and multiplication. Show that if $\varphi(1) \neq 1$, then $\varphi(1)$ is a zero divisor of $R'$. (Zero divisors are defined near the bottom of page 368.)

22. Show that every ideal of the ring $\mathbb{Z}/n\mathbb{Z} = \mathbb{Z}/(n)$ is principle. (Hint: Show every subgroup is cyclic. This can be done quickly using the usual ring map $\pi: \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$, the first test problem and... Another approach is to use the correspondence theorem.)

23. Show that the characteristic of an integral domain is a prime or zero.

24. Let $r$ be a nonzero, nonunit element of an integral domain $R$. Show that $r$ is irreducible if and only if $(r)$ is maximal among all principle ideals of $R$. “$(r)$ is maximal among all principle ideals of $R$” means that whenever $(a)$ is a principle ideal in $R$ satisfying $(r) \subseteq (a) \subseteq R$, either $(a) = R$ or $(a) = (r)$.

25. Show that irreducible elements in a UFD are prime elements.