1. (D&F 0.1.5) Determine whether the following functions \( f \) are well defined:
   
   (a) \( f : \mathbb{Q} \to \mathbb{Z} \) defined by \( f(a/b) = a \).
   
   (b) \( f : \mathbb{Q} \to \mathbb{Q} \) defined by \( f(a/b) = a^2/b^2 \).

2. (D&F 0.1.7+) Let \( f : A \to X \) be a surjective map of sets.
   
   (a) Prove that the relation
       \[
       a \sim b \text{ if and only if } f(a) = f(b)
       \]
       is an equivalence relation (the equivalence classes are called fibers).
   
   (b) Why do we require that \( f \) is surjective?
   
   (c) Describe the fibers when \( f \) is also injective.

3. (D&F 1.1.1-2) Of the following binary operations, determine which are (a) associative, and (b) commutative:
   
   i. the operation defined on \( \mathbb{Z} \) by \( a \star b = a - b \);
   
   ii. the operation defined on \( \mathbb{R} \) by \( a \star b = a + b + ab \).

4. (D&F 1.1.18) Determine (prove positive, or give a reason why not) which of the following sets are groups under addition:
   
   (a) the set of polynomials \( \mathbb{Z}[x] \);
   
   (b) the set of rational numbers (including \( 0 = 0/1 \)) in lowest terms whose denominators are odd;
   
   (c) the set of rational numbers (including \( 0 = 0/1 \)) in lowest terms whose denominators are even;
   
   (d) the set of rational numbers of absolute value < 1;
   
   (e) the set of rational numbers of absolute value \( \geq 1 \) together with 0;
   
   (f) the set of rational numbers (in lowest terms) with denominators equal to 1 or 2;
   
   (g) the set of rational numbers (in lowest terms) with denominators equal to 1, 2, or 3.

5. (D&F 1.1.18) Let \( x, y \in G \). Prove that \( xy = yx \) if and only if \( y^{-1}xy = x \) if and only if \( x^{-1}y^{-1}xy = 1 \).
6. Consider the set of functions

\[ X = \left\{ f_1(x) = x, f_2(x) = \frac{1}{x}, f_3(x) = 1 - x, f_4(x) = \frac{1}{1 - x}, f_5(x) = \frac{-x}{1 - x}, f_6(x) = 1 - \frac{1}{x} \right\}. \]

These functions are all defined on \( \mathbb{R} \setminus \{0,1\} \). Prove \( X \) forms a group where the operation is function composition, and write the group’s multiplication table.

7. (D&F 0.1.11-14) Showing that \( (\mathbb{Z}/n\mathbb{Z})^\times = \{ \bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid (a,n) = 1 \} \) is a group under multiplication:

(a) Prove that if \( \bar{a}, \bar{b} \in \mathbb{Z}^\times \), then \( \bar{a} \cdot \bar{b} \in \mathbb{Z}^\times \).

(b) Let \( n \in \mathbb{Z}, n > 1 \), and let \( a \in \mathbb{Z} \) with \( 1 \leq a \leq n \). Prove that if \( a \) and \( n \) are not relatively prime, there exists an integer \( b \) with \( 1 \leq b < n \) such that \( ab \equiv 0 \) (mod \( n \)) and deduce that there cannot be an integer \( c \) such that \( ac \equiv 1 \) (mod \( n \)).

(c) Let \( n \) and \( a \) be as above. Prove that if \( a \) and \( n \) are relatively prime then there is an integer \( c \) such that \( ac \equiv 1 \) (mod \( n \)) [use the fact that the g.c.d. of two integers is a \( \mathbb{Z} \)-linear combination of the integers].

(d) Conclude from the previous two exercises that \( (\mathbb{Z}/n\mathbb{Z})^\times \) is the set of elements \( \bar{a} \) of \( \mathbb{Z}/n\mathbb{Z} \) with \( (a,n) = 1 \) and hence prove Proposition 4. Verify this directly in the case \( n = 12 \).