Lecture 12
The isomorphism theorems
Picking up from last time

**First isomorphism theorem:** Of $\varphi : G \to H$ is a homomorphism of groups, then $\ker(\varphi) \trianglelefteq G$ and $G/\ker(\varphi) \cong \text{img}(\varphi)$. 
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**First isomorphism theorem:** If \( \varphi : G \to H \) is a homomorphism of groups, then \( \ker(\varphi) \subseteq G \) and \( G/\ker(\varphi) \cong \text{img}(\varphi) \).

Let \( A, B \leq G \), and define \( AB = \{ab \mid a \in A, b \in B\} \).
We showed that \( AB \leq G \) if and only if \( AB = BA \).
We also showed (back in the corollaries to Lagrange’s theorem) that for \( a, a' \in A \),
\[
aB = a'B \text{ if and only if } a(A \cap B) = a'(A \cap B).
\]
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**Theorem (Second (diamond) isomorphism theorem)**

Suppose \( A \leq N_G(B) \) (we say \( A \) normalizes \( B \))

1. Then \( AB \leq G \).

   (In general, if \( B \leq G \), then \( AB \leq G \) for any \( A \leq G \).)
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2. Additionally, \( B \trianglelefteq AB \), \( A \cap B \trianglelefteq A \) and

\[
AB/B \cong A/(A \cap B).
\]
Second (diamond) isomorphism theorem picture
Third isomorphism theorem

Theorem

Let $A, B \trianglelefteq G$ with $A \leq B$. Then

$$A \trianglelefteq B, \quad B/A \trianglelefteq G/A,$$

and

$$\frac{G/A}{B/A} \cong G/B.$$
Theorem
Let $A, B \trianglelefteq G$ with $A \trianglelefteq B$. Then

$$A \trianglelefteq B, \quad B/A \trianglelefteq G/A,$$

and

$$(G/A)/(B/A) \cong G/B.$$ 

Example:

$$(\mathbb{Z}/6\mathbb{Z})/(2\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}.$$
Fourth (lattice) isomorphism theorem

**Theorem**

Let $N \trianglelefteq G$.

There natural projection $\pi : G \to G/N$ gives a bijection

$$\{ A \mid N \leq A \leq G \} \longleftrightarrow \{ \overline{A} \mid \overline{A} \leq G/N \}$$

where $\overline{A} = \pi(A) = A/N$. 
Fourth (lattice) isomorphism theorem

Theorem

Let \( N \trianglelefteq G \).

There natural projection \( \pi : G \to G/N \) gives a bijection

\[
\{ A \mid N \leq A \leq G \} \longleftrightarrow \{ \bar{A} \mid \bar{A} \leq G/N \}
\]

where \( \bar{A} = \pi(A) = A/N \).

For all \( N \leq A, B \leq G \), this bijection additionally satisfies

1. \( A \trianglelefteq G \) if and only if \( \bar{A} \trianglelefteq \bar{G} \),
2. \( A \leq B \) if and only if \( \bar{A} \leq \bar{B} \),
3. if \( A \leq B \), then \( |B : A| = |\bar{B} : \bar{A}| \),
4. \( \langle A, B \rangle = \langle \bar{A}, \bar{B} \rangle \).
Lattice of $D_{16}$

\begin{align*}
\{1, r, r^2, r^3, r^4, r^5, r^6, r^7, s, sr, sr^2, sr^3, sr^4, sr^5, sr^6, sr^7\} \\
\{1, r^2, r^4, r^6, s, sr^2, sr^4, sr^6\} \\
\{1, r^4, sr^2, sr^6\} \\
\{1, sr^6\} \\
\{1\}
\end{align*}
Lattice of $D_{16}$
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Lattice of $D_{16}$

\begin{align*}
\{Z, rZ, r^2Z, r^3Z, sZ, srZ, sr^2Z, sr^3Z\} \\
\{Z, rZ, r^2Z, r^3Z\} \\
\{Z, r^2Z, srZ, sr^3Z\} \\
\{Z, r^2Z, srZ, sr^3Z\} \\
\{Z, sr^2Z\} \\
\{Z, sZ\} \\
\{Z, r^2Z\} \\
\{Z, sr^3Z\} \\
\{Z, srZ\} \\
\{Z, srZ\}
\end{align*}