Lecture 13
The alternating group and more on group actions
Reviewing groups actions

Recall that a group $G$ has a group action on a set $A$

$$\sigma_g : A \to A \quad \text{where} \quad \sigma_g : a \mapsto g \cdot a$$

if

$$1 \cdot a = a \quad \text{and} \quad g \cdot (h \cdot a) = (gh) \cdot a$$

for all $g, h \in G, a \in A$. 

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$$\varphi : G \rightarrow S_A \quad \text{defined by} \quad \varphi : g \rightarrow \sigma_g$$

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Proposition

For any group $G$ and any non-empty set $A$, there is a bijection between

$$
\{ \text{group actions of } G \text{ on } A \} \leftrightarrow \{ \text{homomorphisms } \varphi : G \rightarrow S_A \}. \quad (\text{We showed this bijection in Lecture 5})
$$
The stabilizer of any \( a \in A \) is

\[ G_a = \{ g \in G \mid g \cdot a = a \}, \]

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The kernel of the action is

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\ker = \{g \in G \mid g \cdot a = a \ \text{for all} \ a \in A\} = \bigcap_{a \in A} G_a,
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Proposition

For each \( a \in A \), the number of elements in the equivalence class containing \( a \) is

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|G : G_a|.
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$$a \sim b \quad \text{if and only if} \quad a = g \cdot b \quad \text{for some } g \in G$$

is an equivalence relation.
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**Proposition**

*For each \( a \in A \), the number of elements in the equivalence class containing \( a \) is \( |G : G_a| \).*
Definition
Let $G$ be a group acting on a set $A$.

1. The equivalence class $O_a = \{g \cdot a \mid g \in G\}$ is called the orbit of $a$.

2. The action of $G$ on $A$ is called transitive if there is only one orbit.
Back to the symmetric group

The symmetric group $S_n$ is the group of permutations of $n$ objects.

\[ \sigma = \begin{array}{cccc}
1 & 2 & 3 & 4 \\
\bullet & \bullet & \bullet & \bullet \\
1 & 2 & 3 & 4 \\
\end{array} \quad \begin{array}{ccc}
5 & 6 & 7 \\
\bullet & \bullet & \bullet \\
5 & 6 & 7 \\
\end{array} \]
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We can write a permutation as the product of disjoint cycles:

$$\sigma = (1324)(57)$$
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**Advantage:** unique up to reordering cycles (we still have to prove this!).

**Disadvantage:** How do we write down a presentation of $S_n$?
Reconstructing the symmetric group

**Definition:** The 2-cycles in $S_n$ are called transpositions.

Since

$$(a_1 a_m) \cdots (a_1 a_4)(a_1 a_3)(a_1 a_2) =$$
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every cycle can be written as the product of transpositions, and so every permutation can be written as the product of transpositions. **Caution: not unique!**

For example

$$(1324)(57) = (14)(12)(13)(57)$$
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But

$$S_n = \langle T \rangle \quad \text{where} \quad T = \{(i \ j) \mid 1 \leq i < j \leq n \}.$$
What *is* unique?

\[ S_n = \langle T \rangle \quad \text{where} \quad T = \{(i, j) \mid 1 \leq i < j \leq n\}. \]

**Claim**

Fix \( \sigma \in S_n \). Every expression of \( \sigma \) as a product of transpositions has length of the same parity.
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Claim

Fix \( \sigma \in S_n \). Every expression of \( \sigma \) as a product of transpositions has length of the same parity.

Proof by way of the action on polynomials:

Recall that \( S_n \) acts on \( \mathbb{Z}[x_1, \ldots, x_n] \) by

\[ \sigma \cdot p(x_1, \ldots, x_n) = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}). \]

Let

\[ \Delta = \prod_{1 \leq i < j \leq n} (x_i - x_j) = (x_1 - x_2)(x_1 - x_3) \cdots (x_{n-1} - x_n) \]

\( n(n-1)/2 \) terms
Some examples

\[ \Delta_3 = (x_1 - x_2) (x_1 - x_3) (x_2 - x_3) \]

\[ \Delta_4 = (x_1 - x_2) (x_1 - x_3) (x_1 - x_4) (x_2 - x_3) (x_2 - x_4) (x_3 - x_4) \]

\[ \Delta_5 = (x_1 - x_2) (x_1 - x_3) (x_1 - x_4) (x_1 - x_5) (x_2 - x_3) (x_2 - x_4) (x_2 - x_5) (x_3 - x_4) (x_3 - x_5) (x_4 - x_5) \]
Consider

$$\sigma \cdot \Delta = \prod_{1 \leq i < j \leq n} (x_{\sigma(i)} - x_{\sigma(j)})$$

$$= (-1)^{\#\{1 \leq i < j \leq n \mid \sigma(j) < \sigma(i)\}} \Delta.$$ 

For example, if \(n = 4\) and \(\sigma = (1324)\), then

$$\Delta = (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$

$$(-1)^5 (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_2 - x_3)(x_2 - x_4)(x_3 - x_4)$$
Let

\[ \epsilon = \begin{cases} 
+1 & \text{if } \sigma \cdot \Delta = \Delta, \\
-1 & \text{if } \sigma \cdot \Delta = -\Delta.
\end{cases} \]

Then \( \epsilon(\sigma) \) is the sign of \( \sigma \).
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**Proposition**

*The map \( \epsilon : S_n \to \{\pm 1\}^\times \) is a homomorphism.*
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**Proposition**

The map \( \epsilon : S_n \to \{\pm 1\}^\times \) is a homomorphism.

**Proposition**

All transpositions have negative sign, i.e. \( \epsilon((i \ j)) = -1 \). Therefore,

1. the homomorphism \( \epsilon \) is surjective for all \( n > 1 \), and
2. for any expression of \( \sigma \in S_n \) as the product of transpositions, the parity of the length of that product is determined; namely the length is

   even if \( \epsilon(\sigma) = 1 \), and
   odd if \( \epsilon(\sigma) = -1 \).
Definition
We call a permutation even if $\epsilon(\sigma) = 1$ and odd if $\epsilon(\sigma) = -1$. 

Caution: even/odd
An $m$-cycle is even if $m$ is odd, and is odd if $m$ is even.
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The alternating group $A_n$ is the kernel of $\epsilon : S_n \rightarrow \{\pm 1\}^{\times}$, i.e.

$$A_n = \{\sigma \in S_n \mid \sigma \text{ is even} \}.$$
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Examples

\[ A_1 = \{1\} \]

\[ A_2 = \{1\} \]

\[ A_3 = \{1\} \cup \{3\text{-cycles}\} = \{1, (123), (132)\} \]

\[ A_4 = \{1\} \cup \{3\text{-cycles}\} \cup \{\text{two disjoint 2-cycles}\} \]

\[ = \{1, (123), (124), (132), (134), (142), (143), (234), (243), (12)(34), (13)(24), (14)(23)\} \]