Lecture 6
Special Subgroups
Recall, for any homomorphism $\varphi : G \to H$, the kernel of $\varphi$ is

$$\ker(\varphi) = \{ g \in G \mid \varphi(g) = e_H \} \leq G$$

and the image of $\varphi$ is

$$\text{img}(\varphi) = \{ h \in H \mid \varphi(g) = h \text{ for some } g \in G \} \leq H.$$ 

(They are subgroups of $G$ and $H$ respectively).

**Subgroup criterion:**
A non-empty subset $H \subseteq G$ is a subgroups if and only if

$$xy^{-1} \in H \quad \text{for all } x, y \in H.$$
From last time:

A group action of a group $G$ on a set $A$ is a map from

$$G \times A \rightarrow A$$

$$(g, a) \mapsto g \cdot a$$

which satisfies

$$g \cdot (h \cdot a) = (gh) \cdot a \quad \text{and} \quad 1 \cdot a = a$$

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Any group action is equivalent to a homomorphism

$$\rho : G \rightarrow S_A$$

$$g \mapsto \sigma_g$$

defined by $\rho(g)(a) = \sigma_g(a) = g \cdot a$. 
Group action: \( g \cdot (h \cdot a) = (gh) \cdot a \) and \( 1 \cdot a = a \).

Homomorphism: \( \rho : G \to S_A \), where \( \rho(g)(a) = \sigma_g(a) = g \cdot a \).

**Example:** \( D_8 \) (1) on single vertices, and (2) on unordered pairs of opposite vertices.
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Definition
Let \( a \) be a fixed element of \( A \). The **stabilizer** of \( a \) in \( G \) (with respect to a given action) is

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G_a = \{ g \in G \mid g \cdot a = a \} \subseteq G.
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If \( S \subseteq A \), then

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**Theorem**

*For any non-empty \( S \subseteq A \), \( G_S \) is a subgroup of \( G \).*

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A group acts **on itself** in several ways \((A = G)\). Two important ways are

1. by **left multiplication**: \(g \cdot a = ga\), and
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Example: Let \(D_8\) act on itself by conjugation \((g \cdot a = gag^{-1})\).

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More special subgroups

Definition

Let $A$ be a non-empty subset of $G$ (not nec. subgroup). The centralizer of $A$ in $G$ is $C_G(A) = \{g \in G | gag^{-1} = a \text{ for all } a \in A\}$. Since $gag^{-1} = a \iff ga = ag$, this is the set of elements which commute with all $a$ in $A$. If $A = \{a\}$, we write $C_G(\{a\}) = C_G(a)$. 
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The **center** of a group \( G \), denoted \( Z(G) \), is the set of elements which commute with everything in \( G \), i.e.

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**Corollary**

The center \( Z(G) \) is a subgroup of \( G \) of \( C_G(A) \) for all \( A \subseteq G \).
And one more... Again, let $A \subseteq G$ be a subset of $G$, and fix an element $g \in G$. Let

$$gAg^{-1} = \{ h \in G \mid h = gag^{-1} \text{ for some } a \in A \} \subseteq G$$

be the set of all elements one can arrive at by conjugating elements of $A$ by $g$. Definition

The normalizer of $A$ in $G$ is the set $N_G(A) = \{ g \in G \mid gAg^{-1} = A \} \subseteq G$ of all the elements of $G$ which setwise fix $A$ (individual elements don’t have to be fixed!)

Theorem

For any $A \subseteq G$, the normalizer $N_G(A)$ is a subgroup of $G$. Moreover, $Z(G) \leq C_G(A) \leq N_G(A) \leq G$. 
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