Lecture 9
Quotient groups
Last time

Consider the map

$$\varphi : \mathbb{Z} \to \mathbb{Z}/4\mathbb{Z}$$

$$z \mapsto \bar{z}.$$
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\[ 4\mathbb{Z} = \{4z \mid z \in \mathbb{Z}\} = \{\ldots, -8, -4, 0, 4, 8, \ldots\} \mapsto \{0\} \]

\[ 4\mathbb{Z} + 1 = \{4z + 1 \mid z \in \mathbb{Z}\} = \{\ldots, -7, -3, 1, 5, 9, \ldots\} \mapsto \{1\} \]

\[ 4\mathbb{Z} + 2 = \{4z + 2 \mid z \in \mathbb{Z}\} = \{\ldots, -6, -2, 2, 6, 10, \ldots\} \mapsto \{2\} \]

\[ 4\mathbb{Z} + 3 = \{4z + 3 \mid z \in \mathbb{Z}\} = \{\ldots, -5, -1, 3, 7, 11, \ldots\} \mapsto \{3\} \]

Warm up: Calculate the fibers of the map

\[ \varphi : D_{12} \to S_3 \] defined by \( s \mapsto (12) \) and \( r \mapsto (123). \)
Recall that the **fibers** of a map $\varphi : X \to Y$ are the sets in $\varphi^{-1}(y) \subseteq X$ which all map to the same element $y \in Y$. 

**Theorem**

Let $\varphi : G \to H$ be a surjective homomorphism of groups. For each $h \in H$, let $X_h = \varphi^{-1}(h) = \{ g \in G | \varphi(g) = h \}$ (so, in particular, $X_1 = \text{ker}(\varphi)$).

1. Then $x \in X_a$ and $y \in X_b$ implies $xy \in X_{ab}$. In particular as subsets of $G$, $\{ X_h | h \in H \}$ is a group under the operation $X_a \star X_b = X_{ab}$ (We call this group the **quotient group** $G/\text{ker}(\varphi)$).

2. Fix some fiber $X_h$. For any $x \in X_h$, $X_h = \{ xk | k \in \text{ker}(\varphi) \}$ and $X_h = \{ kx | k \in \text{ker}(\varphi) \}$. 


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1. Then

\[
x \in X_a \text{ and } y \in X_b \quad \text{implies} \quad xy \in X_{ab}.
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In particular as subsets of \( G \), \( \{ X_h \mid h \in H \} \) is a group under the operation \( X_a * X_b = X_{ab} \). (We call this group the quotient group \( G/\ker(\varphi) \)).
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2. Fix some fiber \( X_h \). For any \( x \in X_h \),

\[
X_h = \{xk \mid k \in \ker(\varphi)\} \quad \text{and} \quad X_h = \{kx \mid k \in \ker(\varphi)\}.
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Definition
Let $N \leq G$ and fix $g \in G$. Then

$$gN = \{gx \mid x \in N\} \quad \text{and} \quad Ng = \{xg \mid x \in N\}$$

are the left and right cosets of $N$ in $G$. Every element of $gN$ or $Ng$ is called a representative of the coset.
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**Example:** We just showed that the fibers of a homomorphism are both left and right cosets of the kernel.
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For example, consider $G = S_3$, $N = \{1, (12)\}$, and $g = (23)$. 
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Theorem (Last theorem revisited)
If $K$ is the kernel of some homomorphism of a group $G$, then

$$G/K = \{xK \mid x \in G\}$$

is a group under the multiplication

$$xK \star yK = (xy)K.$$
Skip the homomorphism

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i.e. when is the multiplication

\[ xN \ast yN = (xy)N \] (1)

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**Proposition**

*Let* \( N \leq G \).

1. The left cosets of \( N \) partition the elements of \( G \).
2. \( xN = yN \) if and only if \( y^{-1}x \in N \).
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1. The operation \( \ast \) above is well defined if and only if \( gxg^{-1} \in N \) for all \( x \in N \) and \( g \in G \).
2. If \( \ast \) is well-defined, then \( G/N = \{gN \mid g \in G\} \) forms a group with \( 1 = 1N \) and \( (gN)^{-1} = g^{-1}N \).