Quiz 1, Math 71, Fall 2012

Justify all answers.

(1) Which of the following are groups. Why or why not?
   (a) The positive rational numbers under multiplication.
   (b) The negative rational numbers under multiplication.
   (c) The invertible $2 \times 2$ matrices under the operation

   $$A \star B = B^{-1}A^{-1}.$$

   (a) is, because multiplication is associative, the product of two positive numbers is positive, the multiplicative identity is a positive rational number, and if $a/b$ is positive, then so is $(a/b)^{-1} = b/a.$

   (b) is not because the identity, 1, is not negative.

   (c) is not, because the can’t be an identity: Suppose $A = A \star E = E^{-1}A^{-1}$ Then $E = A^{-2}.$ But that depends on $A!$

(2) Which of the groups from the previous part are also abelian?

   (a) is abelian, and is the only group.
(3) Let $\sigma = (145)(23)$ and $\tau = (15)(24)$ be elements of $S_7$.
(a) What permutation is $\sigma$? In other words, finish the sentence
\[ \sigma \text{ is the map from } [7] \to [7] \text{ which sends } 1 \mapsto \text{??}, 2 \mapsto \text{??}, \ldots \]
(b) Compute $\sigma \tau$.
(c) What is the order of $\sigma$. Why and what does that mean?

\begin{align*}
(a) \ & \sigma \text{ is the map from } [7] \to [7] \text{ which sends} \\
& 1 \mapsto 4, \quad 2 \mapsto 3, \quad 3 \mapsto 2, \quad 4 \mapsto 5, \quad 5 \mapsto 1, \quad 6 \mapsto 6, \quad 7 \mapsto 7. \\
(b) \ & \sigma \tau = (145)(23)(15)(24) = (1)(2543)(6)(7) = (2543). \\
(c) \ & \text{The order of } \sigma \text{ is } \text{lcm}(2, 3) = 6, \text{ which means that } \sigma^6 = 1, \text{ but } \sigma^i \neq 1 \text{ for } 1 < i < 6.
\end{align*}
(4) (a) Define homomorphism of groups.
(b) Define the kernel of a homomorphism.
(c) Verify that 
\[ \varphi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \]
\[ \bar{a} \mapsto \bar{2a} \]
is a homomorphism and give its kernel.
(d) Is \( \varphi \) from part (c) an automorphism? Why or why not?

(1) A homomorphism is a map \( \varphi : G \to H \) satisfying \( \varphi(gg') = \varphi(g)\varphi(g') \) for all \( g, g' \in G \).
(2) The kernel of a homomorphism \( \varphi : G \to H \) is the set \( \{ g \in G \mid \varphi(g) = 1_H \} \).
(3) The map 
\[ \varphi : \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \]
\[ \bar{a} \mapsto \bar{2a} \]
satisfies 
\[ \varphi(\bar{a}) + \varphi(\bar{b}) = 2\bar{a} + 2\bar{b} = 2(\bar{a} + \bar{b}) = \varphi(\bar{a} + \bar{b}) \].
The kernel is \( \{ \bar{a} \in \mathbb{Z}/6\mathbb{Z} \mid 2\bar{a} \equiv \bar{0} \} = \{ \bar{0}, \bar{3} \} \).
(4) The map \( \varphi \) from part (c) is not an automorphism, because the kernel is non-trivial.