Homework 4

1. From an earlier homework, there exists a space X such that \( \pi(X, x_0) = G \). (By the way, this is true for any group \( G \).) Let \( \tilde{X}, \tilde{x}_0 \) be the universal covering space corresponding to \( H = N \leq \pi(X, x_0) \). Then \( \tilde{X} \) is normal and \( \pi(\tilde{X}) \cong G/H \).

Let \( H \leq \pi(X, x_0) \) be the commutator subgroup (which is normal) and \( \tilde{X}_a \) the normal covering corresponding to \( H \). Then \( \pi(\tilde{X}_a) \cong \pi(X, x_0)/H \) is abelian. If \( \tilde{X} \) is a covering such that \( \pi(\tilde{X}) \) is abelian, then \( \pi(X, x_0)/p_* \pi(\tilde{X}, \tilde{x}) \) is abelian. Therefore \( p_* \pi(\tilde{X}_a, \tilde{x}_a) \leq p_* \pi(\tilde{X}, \tilde{x}) \), so there exists a homomorphism \( \phi: \tilde{X}_a \rightarrow \tilde{X} \) such that \( p\phi = p_a \).

2. The subgroups of \( \mathbb{Z} \) are \( n\mathbb{Z} \), \( n \neq 0 \). The subgroups of \( \mathbb{Z}_2 \) are \( \mathbb{Z}_2 \) and \( \{0\} \). We know the covering spaces for each of these subgroups. In the third example, the total space \( X \) is a closed annulus containing a circle as dt. Subgroups of the fundamental group are \( n\mathbb{Z}, n \neq 0 \). The covering spaces (other than \( X \)) are helicoids.

There can be visualized by taking the midpoint of a helical closed line segment \( L \) of length 1 and having it trace out a surface as the center of \( L \) moves around a helix. If the subgroup is \( n\mathbb{Z}, n \neq 1 \), go around the helix \( n \) times and identify the top and bottom (much as was done for the covering \( \mathbb{P} \) of \( S^1 \)).

Define \( \theta: K \rightarrow \pi(X) \) by \( \theta(k) = \psi_k \).

1. \( \psi_k \) is a homeomorphism \( \psi_k(x) = \pi(x, k) = \pi(x) \cdot k = \pi(x) \cdot c = \pi(x) \).
2. \( \theta \) is a homomorphism \( k, l \in K \)
   \[ (\psi_k \psi_l)(x) = \psi_k(kx) = (lk) \cdot x = \psi_{lk}(x) \text{ so } \theta(lk) \theta(k) = \theta(lk) \]
3. \( \theta \) is onto: Let \( \psi \in \pi(X) \) let \( k = \theta(\psi) \). \( \psi_k(x) = \psi(x) \).
   \[ x = \psi_k^{-1} \psi(x) \text{ so } \psi = \theta(k) \].
4. \( \Theta \) is one-one: Suppose \( \Theta(x) = \Theta(y) \Rightarrow x = y \). Apply to \( x \).

(a) This is a 2-sheeted cover. The index of \( \pi \) in \( \pi \) is 2. But any subgroup of index 2 is normal.

(b) Read from left to right and label the vertices \( e_1, e_0, e_1 \) and the medes \( e_1, e_0, e_0, e_1 \). If \( h \in A(\mathbb{E}) \) and \( h \neq \text{id} \), then \( h(e_0) = e_0 \) or \( e_1 \). But at \( e_1 \) there is one loop which projects onto \( B \) and so \( h(e_0) \neq e_0 \). Thus \( e_0 \) and \( e_1 \) have this property.

\( \vdash h = \text{id} \) so \( A(\mathbb{E}) \) is trivial.

5. For \( x_0 \in X \) and define \( \Theta : G \to X \) by \( \Theta(g) = x_0g \). By transitivity, \( \Theta \) is onto. Show \( \Theta(g_1g_2) = \Theta(g_2) \). \( \vdash \Theta \) induces \( \Theta' : G/\langle g_0 \rangle \to X \) which is onto. Show \( \Theta' \) is one-one.