Sketch of Solutions of Homework 9

1. $C_n(K) \approx C_n(A) \otimes C_n(X, M)$ as abelian groups, not as chain complexes. The isomorphism does not induce a homology isomorphism.

Example $(E^n, S^{n-1})$

2. $\chi(X \times Y) = \sum (-1)^n \alpha_n(X \times Y) = \sum \sum (-1)^i \alpha_i(X) \otimes (-1)^j \alpha_j(Y)$

3. $\chi \left( \bigotimes_{m} \alpha(X) \right) \otimes \chi \left( \bigotimes_{m} \beta(Y) \right) = \chi(X) \otimes \chi(Y)$.

4. $\deg f = m$ for some $m$. $f \sim p_m$ where $p_m: S \approx C_m$. $p_m: (1) = 1$.

5. The open cells of $S^1 \times S^2$ are $e_0 \times e_0, e_0 \times e_2, e_0 \times e_2$. This is $S^1 \times S^2 \approx (p+q-1)$-skeleton.

The boundary of $e_0 \times e_0 \times e_2 \subset (p+q-1)$-skeleton.

$S^1 \times S^2 / SPV = \text{closed cell}/\text{its boundary}.$

6. Let $f: \mathbb{R}P^{2m} \to \mathbb{R}P^{2m}$ and $g: S^{2m} \to \mathbb{R}P^{2m}$ be the projections. Since $g$ is a covering map and $\pi(S^{2m}) = 0$, $f$ lifts to $\tilde{f}: S^{2m} \to S^{2m}$, $\tilde{f} = \tilde{g}$.

Let $x_0 \in S^{2m}$ with $\tilde{g}(x_0) = x_0$. $x_0 = -x_0$.

$\tilde{g}(x_0) = [x_0]$. But $\tilde{f}(x_0) = \tilde{g}(x_0) = [x_0] = x_0$.

$x_0$ is a fixed point. Now let $T: \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ be a $LT$ without eigenvalues $T: \mathbb{R}^{2m} \to \mathbb{R}^{2m}$ (otherwise 0 is an eigenvalue).

Let $x, y \in \mathbb{R}^{2m}$ and $x \neq y$. $T = i \chi$. $T(x) = x$.

So $T(x) = x$.

$T$ induces $\tilde{T}: \mathbb{R}P^{2m-1} \to \mathbb{R}P^{2m-1}$.

$x_0$ is a fixed point for $\tilde{T}$, $\tilde{T}(x_0) = x_0$ so $\tilde{T}x = px$.

$p: (p: \mathbb{R}P^{2m-1} \to \mathbb{R}P^{2m-1} \text{ projection})$.

$pTx = px$ so $T(x) = x$.

$f = i \chi$. $T = i \chi$ contradicting the fact that $T$ has no eigenvalues.

7. My the circle onto itself by going around $1/2$ times. $p(t) = (\cos(2 \pi t), \sin(2 \pi t))$, $t \in [0, \frac{1}{2}]$. The map $p$ is onto and null-homotopic ($\deg(p) = 0$). Let $p^n = S^n \times p$ the $n-1$ times suspension of $p$. Then degree $p^n = 0$.
Se it is nullhomotopic. Show \( p^{-1} \) is onto using Massey p. 189.

#8 Use Mayer-Vietoris with \( A_1 = \{(t, b) \mid t > \frac{1}{2}\} \), \( A_2 = \{(t, b) \mid b < \frac{3}{4}\}\).

Then \( A_1 \) and \( A_2 \) are contractible and \( A_1 \cap A_2 \equiv \mathbb{Q} \).