Consider the following argument. Let $X$ be a space and $A \subset X$. The sequence
\[ 0 \to C_n(A) \to C_n(X) \to C_n(X,A) \to 0 \]
of singular chains is exact and $C_n(X,A)$ is free. Therefore
\[ C_n(X) \cong C_n(A) \oplus C_n(X,A). \]
Hence $H_n(X) \cong H_n(A) \oplus H_n(X,A)$.

(a) Show by example that the conclusion above is false.
(b) Find the false assumption and say why it is false.
(c) Give a sufficient condition for the conclusion to hold
(other than $A$ is contractible)

Let $X$ and $Y$ be CW-complexes. Prove
\[ \chi(X \times Y) = \chi(X) \chi(Y). \]
See Massey, top of p. 232 for the product of CW-complexes.

Let $X$ and $Y$ be CW-complexes and $f : X \to Y$ a cellular map
(Massey, p. 232) $f$ induces $f_* : H_n(X^r, X^{r-1}) \to H_n(Y^r, Y^{r-1})$, and hence $f_* : C_r(X) \to C_r(Y)$.
Prove that the following diagram is commutative
\[
\begin{array}{ccc}
H_n(X) & \xrightarrow{\Theta_X} & H_n(X) \\
\downarrow f_* & & \downarrow f_* \\
H_n(Y) & \xrightarrow{\Theta_Y} & H_n(Y)
\end{array}
\]
where $\Theta_X, \Theta_Y$ are the isomorphisms from singular to CW homology.
Hint: Go back to the definition of $\Theta_X, \Theta_Y$.

For every map $f : S^n \to S^n$, show that if map $g : S^n \to S^n$, such that $f \neq g$, and $g$ has a fixed point, $(n \geq 1)$.

Let $S^p$ be the $p$-sphere with CW-structure $S^p = e^0 \cup e^1$. 
and base point $e$. Similarly for $S^8$. Show
\[ S^8 \times S^8 / S^8 \cup S^8 \cong S^{16} \]

(See Massey p. 230 for the product of CW complexes.) Note

If $(X, x_0), (Y, y_0)$ are based spaces then we can take

$X \times Y$ to be $X \times Y$ as $X \times Y$.

Show that every map $\mathbb{R}P^{2n} \to \mathbb{R}P^{2n}$ has a fixed point. (Recall we proved $V_f : S^{2n} \to S^{2n}, \; x \in S^{2n}$ with $f(v) = x + f(x)$

$= -x$. Construct maps $\mathbb{R}P^{2n-1} \to \mathbb{R}P^{2n-1}$ without fixed points

from linear transformations $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ without eigenvalues.

Construct a surjective map $S^n \to S^n$ of degree zero, $n \geq 1$.

(Hint: Try $S^1$ first.)

48.

Compute the homology of the subspace of $I \times I$ consisting of the
four boundary edges plus all the points in the interior whose
first coordinate is rational.