Mathematics 74-114 Midterm Examination - Solutions
Spring 2012

I. Let \( p : \tilde{X} \to X \) be a covering map which is \( n \)-sheeted, \( 2 < n < \infty \). Prove that there is no map \( s : X \to \tilde{X} \) such that \( ps = \text{id} \). (Such a map is called a section of \( p \).)

Proof 1

Let \( x_0 \in X \). Since \( ps = \text{id} \), \( p \circ ps = \text{id} \), and so \( p \) is \( \pi_1(\tilde{X}, \tilde{x}_0) \to \pi_1(X, x_0) \) an onto. Let \( \tilde{x}_0 = s(x_0) \), let \( \tilde{x} \in p^{-1}(x_0) \) and let \( l \) be a path in \( \tilde{X} \) from \( \tilde{x}_0 \) to \( \tilde{x} \). \( [pl] = [p\tilde{x}_0 \tilde{x}l] \) for some loop \( \tilde{m} \) in \( \tilde{X} \) based at \( \tilde{x}_0 \). \( \therefore pl \sim \tilde{m} \) so \( l \sim m \).

\( \tilde{x} = l(1) = m(1) = \tilde{x}_0 \). \( \therefore p^{-1}(x_0) \) has one point. Contradiction.

So \( \exists \) no such \( s \).

Proof 2

\( s : X \to \tilde{X} \) \( p(s) = (ps)p = p \circ p(\text{id}) \), \( \therefore sp \) and id are both lifts of \( p \). To show that they are equal, they must agree on a point. Let \( x_0 \in X \) so \( s(x_0) \in \tilde{X} \).

\( sp(sx_0) = sx_0 = \text{id}(sx_0) \) and so \( sp = \text{id} \). \( \therefore p \) is a homeomorphism so \( \tilde{X} \) is 1-sheeted. Contradiction.
II. Let $G$ be a group with unit $e$ and let $S \subseteq G$ be a set. The normal closure $\overline{S}$ of $S$ is defined to be the intersection of all normal subgroups of $G$ which contain $S$. Prove

$$\overline{S} = \{e\} \cup \{c_1 \cdots c_k \mid k \geq 1, c_i = a_i s_i^{\varepsilon_i} a_i^{-1}, \text{ where } a_i \in G, s_i \in S \text{ and } \varepsilon_i = \pm 1\}.$$ 

Let $H = \langle e \cup \{c_1 \cdots c_k\} \rangle$. Then $H$ is closed under multiplication and inverses and so $H$ is a subgroup. $H$ is normal: Consider

$$x = a_1 c_1 \cdots c_k a_1^{-1} = (a_1 a_1^{-1})(a_2 a_1^{-1}) \cdots (a_k a_1^{-1}).$$

If $c_i = a_i s_i a_i^{-1}$,

then $a c_i = (a a_1 s_i a_1^{-1}) \cdots x \in H$ so $H$ is normal. $H$

contains $S (s_i = e s_i e^{-1})$ so $H$ is a normal subgroup containing $S$. \therefore $\overline{S} \subseteq H$. Conversely, $\overline{S}$ is a normal subgroup containing $S$, so \forall $s_i \in S$, $c_i = a_i s_i a_i^{-1} \in \overline{S}$. \therefore $c_1 c_2 \cdots c_k \in \overline{S} \therefore H \subseteq \overline{S}$.

\therefore $H = \overline{S}$. 

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III. For any two based spaces \((U, u_0)\) and \((V, v_0)\) let \([U, V]\) denote the set of based homotopy classes of based maps \((U, u_0) \to (V, v_0)\). Now let \((A, a_0)\), \((X, x_0)\) and \((Y, y_0)\) be based spaces and define

\[
\theta : [A, X \times Y] \to [A, X] \times [A, Y]
\]

by \(\theta[f] = ([p_1 f], [p_2 f])\), where \(p_1 : X \times Y \to X\) and \(p_2 : X \times Y \to Y\) are the projections. Prove that \(\theta\) is a well-defined bijection.

\[
\text{Suppose } \Theta[fi] = \Theta[gi] \implies p_1 fi = p_2 gi \quad \text{(homotopy)}
\]

\[
\text{and } p_2 fi = p_3 gi \quad \text{(homotopy)}
\]

Then \(fi \simeq gi\) with homotopy \((f_i, g_i)\) \((f_i, g_i)(a) = (f_i(a), g_i(a))\). \(\Rightarrow \theta\) is one-one. If \([h, i] \in [A, X] \times [A, Y]\) and \([k, j] \in [A, X] \times [A, Y]\), then \(h\) and \(k\) determine \((h, k) : A \to X \times Y\)

\[
(h, k)(a) = (h(a), k(a)) \quad \text{and } p_1 (h, k) = h, p_2 (h, k) = k.
\]

\(\Rightarrow \Theta[h, i] = \Theta[k, j]\), so \(\theta\) is onto.
IV. Let \( f : X \to Y \) be a map and let \( p : \tilde{Y} \to Y \) be a covering map. Define the pull-back \( P \) by
\[
P = \{(x, \tilde{y}) \mid x \in X, \tilde{y} \in \tilde{Y} \text{ with } f(x) = p(\tilde{y})\}.
\]
Define maps \( q : P \to X \) and \( r : P \to \tilde{X} \) by \( q(x, \tilde{y}) = x \) and \( r(x, \tilde{y}) = \tilde{y} \).

1. Prove that \( q : P \to X \) is a covering map.
2. Prove that \( r \) induces a bijection \( q^{-1}(x) \to p^{-1}(f(x)) \).
3. Prove that there is a section for \( q : P \to X \) (that is, a map \( s : X \to P \) such that \( qs = \text{id} \)) if and only if \( f \) can be lifted to \( \tilde{Y} \).

1. Let \( x \in X \) and let \( U \) be an elementary nbhd of \( f(x) \).

   Claim: \( f^{-1}(U) \) is an elementary nbhd of \( x \) in \( X \).
   
   \[
p^{-1}(U) = UV, \quad q^{-1}(f^{-1}(U)) = r^{-1}p^{-1}(U) = U r^{-1}(V),
   \]
   a union of disjoint open sets. Clearly \( q' = q |_{p^{-1}(V)} : r^{-1}V \to f^{-1}(U) \). \( q' \) is continuous and we show it is a homeomorphism by constructing an inverse \( k : f^{-1}(U) \to r^{-1}(V) \) defined by
   
   \[
k(x) = (x, (p | V)^{-1}(f(x))).
   \]
   \( q'k(x) = x \) so \( q'k = \text{id} \). Let \( (x, \tilde{y}) \in r^{-1}(V) \subseteq P \)
   
   \[
k'(x, \tilde{y}) = k(x) = (x, (p | V)^{-1}(f(x))).
   \]
   But \( \tilde{y} \in V \)
   
   and \( \tilde{y} = p\tilde{x} \) and so \( \tilde{x} = (p | V)^{-1}p\tilde{x} \) :
   
   \[
k'(x, \tilde{y}) = (x, \tilde{x}).
   \]
   So \( k' \) is a homeomorphism.

2. \( (x, \tilde{y}) \in q^{-1}(x) \), \( x' = p\tilde{y} \) \( r(x, \tilde{y}) = \tilde{x} \in p^{-1}(x) \) : \( r \) induces
   
   \[
r' : q^{-1}(x) \to p^{-1}(x). \quad \text{We define } s' : p^{-1}(x) \to q^{-1}(x) : \text{Given}
   \]
   
   \[
\tilde{y} \in p^{-1}(x), \quad p\tilde{y} = x' \quad \text{so } (x, \tilde{y}) \in P \quad \text{and } \tilde{y} = (x, \tilde{x}) \]
   
   \[
\text{Set } s'(\tilde{y}) = (x, \tilde{x}). \quad \text{Then } r's' = \text{id}, \quad s'r' = \text{id} \quad \text{so } r' \text{ is bijection.}
   \]

3. \( \text{If } s \text{ is a section for } q, \text{ then } r's \text{ is a lift of } f \text{ to } \tilde{Y} \).

   Conversely, if \( \tilde{y} \) is a lift of \( f \) to \( \tilde{Y} \), define \( s : X \to P \) by
   
   \[
s(x) = (x, \tilde{y}(x)).
   \]
V. Let $\tilde{X}$ be any normal cover of $X$ with covering map $p$, let $x_0 \in X$ be the base point and choose $\tilde{x}_0 \in p^{-1}(x_0)$. Define $\theta : \pi(X, x_0) \to A(\tilde{X})$ (the group of deck transformations) as follows: Let $\alpha = [l] \in \pi(X, x_0)$ and let $\tilde{l}$ be the lift of $l$ to $\tilde{X}$ starting at $\tilde{x}_0$. Set $x_0' = \tilde{l}(1)$. Then $p_*\pi(\tilde{X}, \tilde{x}_0)$ and $p_*\pi(\tilde{X}, x_0')$ are conjugate, hence equal. Therefore there exists $\phi \in A(\tilde{X})$ with $\phi(\tilde{x}_0) = x_0'$. Set $\theta(\alpha) = \phi$. Prove

1. $\theta$ is a homomorphism.
2. Kernel $\ker \theta = p_*\pi(\tilde{X}, \tilde{x}_0)$.

Thus $\theta$ incuces a homomorphism $\theta' : \pi(X, x_0)/p_*\pi(\tilde{X}, \tilde{x}_0) \to A(\tilde{X})$, where $\pi(X, x_0)/p_*\pi(\tilde{X}, \tilde{x}_0)$ is the set of right cosets. Prove

3. $\lambda\theta' = \mu$, where $\lambda : A(\tilde{X}) \to p^{-1}(x_0)$ and $\mu : \pi(X, x_0)/p_*\pi(\tilde{X}, \tilde{x}_0) \to p^{-1}(x_0)$ have been defined in class.

\[
\begin{align*}
1. \beta = \sum J, \text{ lift of } m \text{ starting at } \tilde{x}_0. \text{ Let } \psi \in A(\tilde{X}) \text{ such that } \\
\psi(\tilde{x}_0) = \tilde{x}_0(1). \text{ Then } \psi \text{ is lift of } m \text{ starting at } \psi(\tilde{x}_0) = x_0' \\
\text{ Here path } \tilde{\gamma} \cdot \psi \tilde{\gamma} \text{ in } \tilde{X} \text{ starting at } \tilde{x}_0 \text{ and } p(\tilde{\gamma} \cdot \psi \tilde{\gamma}) = m \\
\theta(\tilde{\gamma}) \in A(\tilde{X}) \text{ and } \theta(\tilde{\gamma}) (\tilde{x}_0) = (\gamma \cdot \psi \tilde{\gamma})(1) = \psi(\tilde{\gamma}1). \\
\theta(\tilde{\gamma}) \psi(\tilde{\gamma}) = \psi \in A(\tilde{X}) \text{ and } \psi(\tilde{x}_0) = \psi(\tilde{x}_0) \Rightarrow \theta(\tilde{\gamma}) = \theta(\tilde{\gamma}) (\tilde{x}_0). \\
\text{ Let } \gamma = \sum K \in \ker \theta, \quad \theta(\gamma) = m. \text{ Let } \tilde{k} \text{ be a lift of } k \text{ starting at } \tilde{x}_0, \quad \tilde{k}(\tilde{x}_0) = \tilde{x}_0 = \tilde{1}(1), \text{ so } \tilde{k} \text{ is a loop, } \sum K \in \pi'(\tilde{X}, \tilde{x}_0). \\
p_k(\sum K) = \gamma \text{ so } \gamma \in \Im p_k. \text{ Conversely, if } \gamma = \sum K \text{ then } \gamma \cdot \tilde{k} \text{ is a loop in } \tilde{X} \text{ based at } \tilde{x}_0, \text{ m is } \tilde{k} \text{ a lift of } k. \text{ If } \theta(\gamma) = \psi, \quad \psi(\tilde{x}_0) = m(1) = \tilde{x}_0. \text{ Then } \psi = m \text{ so } \gamma \in \ker \theta.
\end{align*}
\]

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Consider the diagram
\[ \xymatrix{ \pi(X, x_0) \ar[r]^{\theta} & A(\tilde{x}) \ar[d]^{p^{-1}(\tilde{x}_0)} \ar[ld]^{\tilde{\mu}} } \]

If \( \alpha \in \pi(X, x_0) \), \( \alpha = \tilde{\epsilon} \) and \( \tilde{\epsilon} \) is a lifting of \( \epsilon \) starting at \( \tilde{x}_0 \), \( \tilde{\mu}(\alpha) = \tilde{\epsilon}(1) = x_0 \). Then
\[ \lambda \theta(\alpha) = \lambda(\epsilon) = \psi(\tilde{x}_0), \]
where \( \psi \in A(\tilde{x}) \), such that \( \psi(\tilde{x}_0) = \tilde{\epsilon}(1) = x_0 \).

\[ \therefore \lambda \theta(\alpha) = \psi(\tilde{x}_0) = \tilde{x}_0 = \tilde{\mu}(\alpha) \]

\[ \therefore \text{The diagram is commutative: } \lambda \theta = \tilde{\mu}. \]

Let \( \nu : \pi(X, x_0) \rightarrow \pi(X, \tilde{x}_0) \) be the quotient map \( \tilde{\mu} \text{ mod } \mu \nu \) and
\[ \theta' \nu = \theta. \]

Moreover, \( \lambda \theta = \tilde{\mu} \) becomes
\[ \lambda \theta' \nu = \mu \nu. \]
Since \( \nu \) is onto, \( \lambda \theta' = \tilde{\mu} \).