Exercises

1. Suppose $F$ is a finite field with $q$ elements and let $f(x) \in F[x]$ have degree $d$. Show that $f$ is irreducible if and only if the monic gcd of $f(x)$ and $x^{q^j} - x$ is 1 for each $j < d$.

2. Suppose $F$ is a finite field with $q$ elements and let $f(x) \in F[x]$ have degree $d$. Show that $f$ is irreducible if and only if $f(x) \mid x^{q^d} - x$ and the monic gcd of $f(x)$ and $x^{q^j} - x$ is 1 for each $j < d$ with $j \mid d$.

3. Suppose $p$ is a prime and $F$ is a finite field with $p^{15}$ elements. We know that the polynomial $x^{p^{15}} - x$ has every element of $F$ as a root. Let $K$ be the set of roots in $F$ of the polynomial $x^{p^3} - x$. Show that $K$ is a subfield of $F$.

4. Let $F = \mathbb{Z}/(2)$. Set $K = F[x]/(x^3 + x + 1)$ and $L = F[x]/(x^3 + x^2 + 1)$. Then both $K$ and $L$ are 8-element fields, so must be isomorphic. Find an isomorphism from $K$ to $L$. 