1. (10) Determine if the improper integral

\[ \int_0^\infty \frac{x}{1+x^2} \, dx \]

converges or diverges.

Let \( u = 1 + x^2 \), then \( du = 2x \, dx \)

The indefinite integral is

\[ \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln(u) \]

To evaluate, take

\[ \lim_{t \to \infty} \left[ \frac{1}{2} \ln(1+x^2) \right]_0^t \]

\[ = \lim_{t \to \infty} \left( \frac{1}{2} \ln(1+t^2) - \frac{1}{2} \ln(1) \right) \]

\[ = \frac{1}{2} \ln(1+t^2) \]

Since \( 1+t^2 \to \infty \) as \( t \to \infty \),

\[ \ln(1+t^2) \to \infty \] as \( t \to \infty \)

The limit does not exist, so the integral diverges.
2. Determine if the series
\[ \sum_{n=2}^{\infty} \frac{\sqrt{2n}}{n^2 - 1} \]
converges. Mention any test that you might use and verify that it is applicable.

All terms are positive. \( \frac{n^{1/2}}{n^2} \) is like \( \frac{1}{n^{3/2}} \) which gives a convergent p-series (also with all positive terms) so I will use limit comparison.

\[
\lim_{n \to \infty} \frac{\sqrt{2n}}{1/n^{3/2}} = \lim_{n \to \infty} \frac{\sqrt{2} \cdot n^{1/2} \cdot n^{3/2}}{n^2 - 1} = \lim_{n \to \infty} \frac{\sqrt{2} \cdot n^2}{n^2 - 1} = \sqrt{2}
\]

Since \( 0 < \sqrt{2} < \infty \) the limit test shows \( \sum_{n=2}^{\infty} \frac{\sqrt{2n}}{n^2 - 1} \)
and \( \sum_{n=2}^{\infty} \frac{1}{n^{3/2}} \) have the same convergence properties. Since the latter is a convergent p-series, the former (given) series converges.
3. (14) The following power series has radius of convergence \( R = 7 \).

\[
\sum_{n=1}^{\infty} \frac{(x - 2)^n}{\sqrt[n]{n} 7^n}
\]

Find the interval of convergence. Mention any test that you might use and verify that it is applicable.

We need to check the endpoints of \((2-7, 2+7)\).

\(x = -5\):

\[
\sum_{n=1}^{\infty} \frac{(-7)^n}{\sqrt[n]{n} 7^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[n]{n}}
\]

Since \(\frac{1}{\sqrt[n]{n}}\) decreases to zero, the Alternating Series Test shows this series converges.

\(x = 9\):

\[
\sum_{n=1}^{\infty} \frac{(7)^n}{\sqrt[n]{n} 7^n} = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}
\]

This is a convergent \(p\)-series.

Hence the interval of convergence is \([-5, 9)\).
4. (12) Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{nx^n}{(n+1)!2^{n-1}}$$

**Ratio Test**

$$|a_{n+1}| = \frac{(n+1)|x|^{n+1}}{(n+2)! \cdot 2^n}$$

$$|a_n| = \frac{n|x|^n}{(n+1)! \cdot 2^{n-1}}$$

$$\left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)|x|^{n+1}(n+1)!2^{n-1}}{(n+2)!2^n(n)|x|^n} = \frac{(n+1)|x|}{2n(n+2)}$$

$$= \left| \frac{x(n+1)}{2n^2 + 4n} \right|$$

$$\lim_{n \to \infty} \left| \frac{x(n+1)}{2n^2 + 4n} \right| = 0 \quad \text{independent of the value of } x$$

so the radius of convergence is $\infty$. 
5. (10) Suppose that \( f(x) \) is equal to its Taylor series

\[
f(x) = \sum_{n=0}^{\infty} \frac{1}{2^n} (x - 3)^n
\]

about \( a = 3 \). What is the 39th derivative \( f^{(39)}(3) \)? You need not simplify your answer. No partial credit will be given for this problem.

The term \( n = 39 \) of the Taylor series is

\[
\frac{f^{(39)}(3)}{39!} (x-3)^{39}
\]

from the definition

\[\text{and } \frac{1}{2^{39}} (x-3)^{39} \text{ from the equation given here.}\]

Thus \( f^{(39)}(3) = \frac{39!}{2^{39}} \).
6. (12) Write down the first three non-zero terms of the Taylor series for \( \ln(2x + 4) \) at \( a = 1 \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( f^{(n)}(x) )</th>
<th>( f^{(n)}(1) )</th>
<th>( n! )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>( \ln(2x + 4) )</td>
<td>( \ln 6 )</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( \frac{1}{x + 2} )</td>
<td>( \frac{1}{3} )</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>( \frac{-1}{(x + 2)^2} )</td>
<td>( -\frac{1}{9} )</td>
<td>2</td>
</tr>
</tbody>
</table>

\[ \ln 6 + \frac{1}{3}(x-1) - \frac{1}{18}(x-1)^2 \]
7. (12) Express the integral
\[ \int 2(2 + x)^{-1} \, dx \]
as a MacLaurin series. It suffices to write down the first four non-zero terms. You may assume that the arbitrary constant \( C = 0 \).

\[
\frac{2}{2 + x} = \frac{2}{2(1 + \frac{x}{2})} = \frac{1}{1 + \frac{x}{2}} = \sum_{n=0}^{\infty} (-\frac{x}{2})^n
\]

\[
\int \sum_{n=0}^{\infty} (-\frac{x}{2})^n \, dx = \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n x^n \, dx
\]

\[
= \sum_{n=0}^{\infty} \left( -\frac{1}{2} \right)^n \frac{x^{n+1}}{n+1} \quad (+ \text{constant})
\]

First four terms:
\[
x - \frac{x^2}{4} + \frac{x^3}{12} - \frac{x^4}{32}
\]
8. (18) For each of the following statements, fill in the blank with the letters T or F depending on whether the statement is true or false. You do not need to show your work and no partial credit will be given on this problem.

(a) The sequence \( \left\{ \left( \frac{\pi}{3} \right)^n \right\} \) converges.

\[
\frac{\pi}{3} > 1 \quad \text{so} \quad n \to \infty
\]

ANS: \( F \)

(b) The series \( \sum_{n=1}^{\infty} a_n \) converges if and only if \( \lim_{n \to \infty} (a_1 + a_2 + \cdots + a_n) \) exists.

\[
\lim_{n \to \infty} \text{(partial sums)} \quad \text{needs to be finite}
\]

(Def. of convergence for series)

ANS: \( T \)

(c) The series \( .9 + .99 + .999 + \cdots \) converges to 1.

The sequence \( .9, .99, .999, \ldots \) converges to 1, but that means this series diverges by the Test for Divergence.

ANS: \( F \)
(d) If $\sum_{n=1}^{\infty} a_n$ is a divergent series, then $\sum_{n=1}^{\infty} |a_n|$ is a divergent series.

This is equivalent to the statement

"if $\sum a_n$ converges absolutely, then it converges" (contrapositive)

ANS: $\mathbf{T}$

(e) $\lim_{n \to \infty} \left( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \right) = 1$.

geometric $\sum_{n=0}^{\infty} \frac{1}{2} \left( \frac{1}{2} \right)^n$

since $\frac{1}{2} < 1$, conv. to $\frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$

ANS: $\mathbf{T}$

(f) If $0 \leq a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

exactly the statement of the (direct)

comparison test.

ANS: $\mathbf{T}$