Lagrange Multipliers

Page 793: Solutions to Problems 1, 3, 5, 9 and 11

1. Use the method of Lagrange multipliers to maximize x^3y^5 subject to the constraint $x + y = 8$.

Answer: Let $f(x, y) = x^3y^5$ and $g(x, y) = x + y - 8$. The points (x, y) which will maximize $f(x, y)$ subject to the constraint $g(x, y) = 0$ will be the set of points (x, y) that satisfy the equations $\nabla f(x, y) = \lambda \nabla g(x, y)$ and $g(x, y) = 0$. Writing out the system of three equations and three unknowns explicitly:

$$\begin{aligned} 3x^2y^5 &= \lambda & (1) \\ 5x^3y^4 &= \lambda & (2) \\ x + y &= 8. & (3) \end{aligned}$$

Suppose $\lambda = 0$. Then eqn.(1) would be $3x^2y^5 = 0$ which would imply that either $x = 0$ or $y = 0$. (They cannot both be equal to 0, because that would contradict eqn.(3).) If $x = 0$, then eqn.(3) says $y = 8$. And if $y = 0$, we would instead have from eqn.(3) that $x = 8$. Either way, we get $f(8, 0) = f(0, 8) = 0$.

Now suppose that $\lambda \neq 0$. We can set equations (1) and (2) equal to each other and simplify:

$$\begin{aligned} 3x^2y^5 &= 5x^3y^4 \\ 3x^2y^5 - 5x^3y^4 &= 0 \\ x^2y^4(3y - 5x) &= 0 & (4) \end{aligned}$$

Since $\lambda \neq 0$, eqn.(1) implies that $x \neq 0$, $y \neq 0$. Therefore, for eqn.(4) to be true, it must be that $3y - 5x = 0$, and solving this equation for y: $y = \frac{5}{3}x$. Plugging this into eqn.(3) and solving for x yields:

$$x + \frac{5}{3}x = 8 \Longrightarrow \frac{8}{3}x = 8 \Longrightarrow x = 3 \text{ and therefore } y = 5.$$

And so we have $f(3, 5) = 3^3 \cdot 5^5 = (27) \cdot (3, 125) = 84, 375$. Since clearly $f(3, 5) > f(8, 0)$ and $f(0, 8)$, $f(x, y)$ is maximized at the point $(3, 5)$.

3. Find the distance from the origin to the plane $x + 2y + 2z = 3$,

(i) using a geometric argument (no calculus)

(ii) by reducing the problem to an unconstrained problem in two variables, and

(iii) using the method of Lagrange multipliers.

Answer:
(i): From Example 7 on page 628, we know that the distance s from the point (x_0, y_0, z_0) to the plane whose equation is $Ax + By + Cz = D$ is given by the formula:

$$s = \frac{|Ax_0 + By_0 + Cz_0 - D|}{\sqrt{A^2 + B^2 + C^2}}.$$

And so, plugging in the appropriate numbers:

$$s = \frac{|(1 \cdot 0) + (2 \cdot 0) + (2 \cdot 0) - 3|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{3}{\sqrt{9}} = \frac{3}{3} = 1.$$

(ii): The distance from any point on the plane to the origin is $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$. This is the function I wish to minimize. To make Life (and taking derivatives) easier, I will instead minimize its square: $F(x, y, z) = x^2 + y^2 + z^2$. The point (x, y, z) that minimizes $F(x, y, z)$ will also certainly minimize $f(x, y, z)$.

Let's solve the equation of the plane for z: $z = \frac{1}{2}(3 - x - 2y)$, and then plug this z into $F(x, y, z)$:

$$\begin{aligned} F(x,y,z) &= F\left(x, y, \frac{1}{2}(3-x-2y)\right) \\ &= x^2 + y^2 + \left(\frac{1}{2}(3-x-2y)\right)^2 \\ &= x^2 + y^2 + \frac{1}{4}(3-x-2y)^2 \end{aligned}$$

Let that last equation equal $g(x, y)$. I now need to find the critical points of $g(x, y)$. Once I find the critical points, I plug them into $g(x, y)$ and the smallest number I get will be the square of the minimum distance. (Remember, I'm minimizing the distance-squared function.)

To find the critical points, I have to find the (x, y) such that $\nabla g(x, y) = 0\mathbf{i} + 0\mathbf{j}$. So the equations I need to solve are

$$\begin{aligned} g_x(x,y) &= -\frac{3}{2} + \frac{5}{2}x + y = 0 \\ g_y(x,y) &= -3 + x + 4y = 0. \end{aligned}$$

Solving the above for x and y: $x = \frac{1}{3}$ and $y = \frac{2}{3}$, and then we get $g\left(\frac{1}{3}, \frac{2}{3}\right) = 1$, and so the distance is 1.

(iii): As in part (ii), I'll instead minimize the distance-squared: $F(x, y, z) = x^2 + y^2 + z^2$. The constraint I have is $g(x, y, z) = x + 2y + 2z - 3 = 0$, and this means the system of equations I need to solve is

$$\begin{aligned} 2x &= \lambda && (5) \\ 2y &= 2\lambda && (6) \\ 2z &= 2\lambda && (7) \\ x + 2y + 2z &= 3 && (8) \end{aligned}$$

Suppose $\lambda = 0$. Then eqns.(5-7) would imply $x = y = z = 0$. But then that would contradict eqn.(8): $0 + 0 + 0 = 3$. Therefore, we must have $\lambda \neq 0$.

Since $\lambda \neq 0$, setting eqn.(6) equal to eqn.(7) yields: $2y = 2z \Longrightarrow y = z$. (Note that to get this, I have to divide by λ. Since I know $\lambda \neq 0$, I can legally do so.) And using eqns.(6) and (5), I see that $y = 2x$. Substituting all this into eqn.(8) and solving:

$$x + 2(2x) + 2(2x) = 3 \Longrightarrow 9x = 3 \Longrightarrow x = \frac{1}{3},$$

and so $y = \frac{2}{3}$ and $z = \frac{2}{3}$. And so (a big surprise here): $F\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right) = 1$ and that means the minimum distance is 1.

5. Use the Lagrange multiplier method to find the greatest and least distances from the point $(2, 1, -2)$ to the sphere with the equation $x^2 + y^2 + z^2 = 1$.

Answer: The distance from (x, y, z) to the point $(2, 1, -2)$ is $f(x, y, z) = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$. This is the function I need to minimize and maximize subject to the constraint that $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Using the same reasoning given in the previous problem, I will instead minimize and maximize $F(x, y, z) = (x-2)^2 + (y-1)^2 + (z+2)^2$.

Taking the appropriate derivatives, the system of equations I need to solve is

$$\begin{aligned} 2(x-2) &= 2x\lambda && \text{which simplifies to:} && x(1-\lambda) = 2 && (9) \\ 2(y-1) &= 2y\lambda && \text{which simplifies to:} && y(1-\lambda) = 1 && (10) \\ 2(z+2) &= 2z\lambda && \text{which simplifies to:} && z(1-\lambda) = -2 && (11) \\ x^2 + y^2 + z^2 &= 1 && \text{which stays the same:} && x^2 + y^2 + z^2 = 1. && (12) \end{aligned}$$

Now, eqns.(9-11) all imply that $x, y, z \neq 0$. (Otherwise, we would have $0 = 2$.)

Since I know that $x, y, z \neq 0$, I can legally divide both sides of eqn.(9) by x, both sides of eqn.(10) by y and both sides of eqn.(11) by z. (Before dividing, I need to first make sure that I'm not dividing by 0.) And so

$$\begin{aligned} 1-\lambda &= \frac{2}{x} && (13) \\ 1-\lambda &= \frac{1}{y} && (14) \\ 1-\lambda &= -\frac{2}{z} && (15) \end{aligned}$$

Setting eqns.(13) and (15) equal to each other gets:

$$\frac{2}{x} = -\frac{2}{z} \Longrightarrow z = -x$$

and from eqns.(13) and (14):

$$\frac{2}{x} = \frac{1}{y} \Longrightarrow y = \frac{x}{2}.$$

Let's plug all this into eqn.(12):

$$x^2 + \left(\frac{x}{2}\right)^2 + (-x)^2 = 1 \Longrightarrow \frac{9}{4}x^2 = 1 \Longrightarrow x^2 = \frac{4}{9} \Longrightarrow x = \pm\frac{2}{3}.$$

The two points we need to check are $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ and $\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$:

$$\begin{aligned} F\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) &= 4 \\ F\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) &= 16. \end{aligned}$$

Therefore, the least distance from the point $(2, 1, -2)$ to the unit sphere centered at the origin is $\sqrt{4} = 2$, and the greatest distance is $\sqrt{16} = 4$.

9. Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$.

Answer: The function we want to maximize and minimize is $f(x, y, z) = xyz$, and the constraint we have is $g(x, y, z) = x^2 + y^2 + z^2 - 12 = 0$. The system of equations we need to solve are

$$\begin{aligned} yz &= 2x\lambda && (16) \\ xz &= 2y\lambda && (17) \\ xy &= 2z\lambda && (18) \\ x^2 + y^2 + z^2 &= 12. && (19) \end{aligned}$$

Suppose that $\lambda = 0$. Then exactly *two* variables must be equal to 0. (E.g. Suppose $x = 0$ and $y, z \neq 0$. Then we'd get a contradiction from eqn.(16): $yz = 0$.) And they can't all be 0, because that would contradict eqn.(19). So no matter which variable is not equal to 0, we would have

$$f(\pm\sqrt{12}, 0, 0) = f(0, \pm\sqrt{12}, 0) = f(0, 0, \pm\sqrt{12}) = 0.$$

The $\pm\sqrt{12}$ comes from solving eqn.(19).

Now suppose that $\lambda \neq 0$. Let me multiply both sides of eqn.(16) by x, both sides of eqn.(17) by y, and both sides of eqn.(18) by z, and add those three equations together:

$$\begin{aligned} 3xyz &= 2\lambda(x^2 + y^2 + z^2) \\ 3xyz &= 2\lambda(12) \\ xyz &= 8\lambda. \end{aligned} \tag{20}$$

That last equation implies that $x, y, z \neq 0$ (since $\lambda \neq 0$) . Let me use eqn.(18) in eqn.(20) and solve for z:

$$\begin{aligned} xyz &= 8\lambda \\ (xy)z &= 8\lambda \\ (2z\lambda)z &= 8\lambda \\ 2z^2 &= 8 \\ z^2 &= 4 \\ z &= \pm 2. \end{aligned}$$

I can divide by λ in that fourth line because $\lambda \neq 0$.

Similarly, I can use eqn.(17) in eqn.(20) and solve for y: $y = \pm 2$. And with eqn.(16) in eqn.(20), I can get x: $x = \pm 2$. And so there are a few possibilities. Let's make a table of the different combinations:

x	y	z	$f(x, y, z)$
2	2	2	8
2	2	-2	-8
2	-2	2	-8
2	-2	-2	8
-2	2	2	-8
-2	2	-2	8
-2	-2	2	8
-2	-2	-2	-8

Comparing all the numbers (e.g. -8, 0 and 8), we see that the minimum value of $f(x, y, z)$ is -8, and the maximum value is 8.

11. Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

Answer: So we have to minimize and maximize the function $f(x, y, z) = x$ subject to the two constraints $g(x, y, z) = x + y - z = 0$ and $h(x, y, z) = x^2 + 2y^2 + 2z^2 - 8 = 0$. Therefore, the equations we have to solve are $\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$. That means, we solve

$$1 = \lambda + 2x\mu \tag{21}$$
$$0 = \lambda + 4y\mu \tag{22}$$
$$0 = -\lambda + 4z\mu \tag{23}$$
$$x + y = z \tag{24}$$
$$x^2 + 2y^2 + 2z^2 = 8. \tag{25}$$

Note that it must be the case that $\mu \neq 0$. (If $\mu = 0$, then eqn.(21) would say $\lambda = 1$ and eqn.(22) would say $\lambda = 0$, and we would get a contradiction.)

Since I know I can legally divide by μ, from eqns.(22) and (23) we have

$$4y\mu = -4z\mu \Longrightarrow y = -z$$

and using that in eqn.(24):

$$x + (-z) = z \Longrightarrow x = 2z.$$

Let's plug all this into eqn.(25):

$$
\begin{aligned}
(2z)^2 + 2(-z)^2 + 2z^2 &= 8 \\
4z^2 + 2z^2 + 2z^2 &= 8 \\
8z^2 &= 8 \\
z^2 &= 1 \\
z &= \pm 1.
\end{aligned}
$$

For $z = 1$, we have $x = 2$ and $y = -1$, and $f(2, -1, 1) = 2$. For $z = -1$, we have $x = -2$ and $y = 1$, and $f(-2, 1, -1) = -2$. Therefore, the minimum value is -2, and the maximum value 2.

Figure 1: Joseph-Louis Lagrange (1736-1813)