Functions of Several Variables

Lecture 21

November 6, 2006
Functions of two variables

A function of two variables is a rule that assigns to each ordered pair of real numbers \((x, y)\) in a set \(D\) a unique real number denoted by \(f(x, y)\). The set \(D\) is the domain of \(f\) and its range is the set of values that \(f\) takes on.

We also write \(z = f(x, y)\). The variables \(x\) and \(y\) are independent variables and \(z\) is the dependent variable.
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The variables \(x\) and \(y\) are independent variables and \(z\) is the dependent variable.
Examples

Find the domain of the function \( f(x, y) = 2x + 3y \). Find the domain and range of \( f(x, y) = \sqrt{4 - x^2 - y^2} \).
Examples

Find the domain of the function

\[ f(x, y) = \frac{2x + 3y}{x^2 + y^2 - 9} \]
Examples

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\[ f(x, y) = \frac{2x + 3y}{x^2 + y^2 - 9} \]

- Find the domain and range of

\[ f(x, y) = \sqrt{4 - x^2 - y^2} \]
Definition

If \( f \) is a function of two variables with domain \( D \), then the graph of \( f \) is the set of all points \((x, y, z) \in \mathbb{R}^3\) such that \( z = f(x, y) \) and \((x, y)\) is in \( D \).
A linear function is a function $f(x) = ax + by + c$. The graph of such a function is a plane.
Example

A linear function is a function

\[ f(x) = ax + by + c \]
Example

- A **linear function** is a function
  
  \[ f(x) = ax + by + c \]
  
- The graph of such a function is a plane.
Examples

Example

- $f(x, y) = \sin(x) + \sin(y)$
Example

- \( f(x, y) = \sin(x) + \sin(y) \)
Example

\[ f(x, y) = (x^2 + y^2)e^{-x^2 - y^2} \]
Example

\[ f(x, y) = (x^2 + y^2)e^{-x^2 - y^2} \]
The Cobb-Douglas production function

Example

\[ P(L, K) = bL^\alpha K^{1-\alpha} \]
The Cobb-Douglas production function

Example

- \( P(L, K) = bL^\alpha K^{1-\alpha} \)
- \( P(L, K) = 1.01L^{0.75}K^{0.25} \)
Level Curves

Definition

The **level curves** of a function $f$ of two variables are the curves with equations $f(x, y) = k$, where $k$ is constant.
### Example

- \( f(x, y) = \sin(x) + \sin(y) \)
Example

- \( f(x, y) = \sin(x) + \sin(y) \)
Example

\[ f(x, y) = (x^2 + y^2)e^{-x^2 - y^2} \]
The Cobb-Douglas production function

Example

\[ P(L, K) = 1.01L^{0.75}K^{0.25} \]
Limits and Continuity

Definition

We say that a function \( f(x, y) \) has limit \( L \) as \((x, y)\) approaches a point \((a, b)\) and we write

\[
\lim_{(x, y) \to (a, b)} f(x, y) = L
\]

if we can make the values of \( f(x, y) \) as close to \( L \) as we like by taking the point \((x, y)\) sufficiently close to the point \((a, b)\), but not equal to \((a, b)\).

We write also

\[
f(x, y) \to L \quad \text{as} \quad (x, y) \to (a, b)
\]

and

\[
\lim_{x \to a, y \to b} f(x, y) = L
\]
We say that a function \( f(x, y) \) has limit \( L \) as \( (x, y) \) approaches a point \( (a, b) \) and we write

\[
\lim_{(x,y) \to (a,b)} f(x, y) = L
\]

if we can make the values of \( f(x, y) \) as close to \( L \) as we like by taking the point \( (x, y) \) sufficiently close to the point \( (a, b) \), but not equal to \( (a, b) \).
Limits and Continuity

**Definition**

- We say that a function $f(x, y)$ has limit $L$ as $(x, y)$ approaches a point $(a, b)$ and we write
  
  $$\lim_{(x,y) \to (a,b)} f(x, y) = L$$

  if we can make the values of $f(x, y)$ as close to $L$ as we like by taking the point $(x, y)$ sufficiently close to the point $(a, b)$, but not equal to $(a, b)$.

- We write also $f(x, y) \to L$ as $(x, y) \to (a, b)$ and
  
  $$\lim_{x \to a, y \to b} f(x, y) = L$$

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Important

If \( f(x, y) \to L_1 \) as \((x, y) \to (a, b)\) along a path \(C_1\) and \( f(x, y) \to L_2 \) as \((x, y) \to (a, b)\) along a path \(C_2\), where \(L_1 \neq L_2\), then \( \lim_{(x, y) \to (a, b)} f(x, y) \) does not exist.

Example: Show that \( \lim_{(x, y) \to (0, 0)} x^2 - y^2 \) does not exist.
If \( f(x, y) \to L_1 \) as \((x, y) \to (a, b)\) along a path \( C_1 \) and \( f(x, y) \to L_2 \) as \((x, y) \to (a, b)\) along a path \( C_2 \), where \( L_1 \neq L_2 \), then \( \lim_{(x,y)\to(a,b)} f(x, y) \) does not exist.
Important

If \( f(x, y) \to L_1 \) as \( (x, y) \to (a, b) \) along a path \( C_1 \) and 
\( f(x, y) \to L_2 \) as \( (x, y) \to (a, b) \) along a path \( C_2 \), where 
\( L_1 \neq L_2 \), then \( \lim_{(x,y)\to(a,b)} f(x, y) \) does not exists.

**Example:** Show that 
\[
\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}
\]
does not exist.
A function $f$ of two variables is called continuous at $(a, b)$ if
$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b).$$

Examples: polynomials, rational, trigonometric, exponential, logarithmic functions are continuous on their domain.
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\textbf{Examples:} polynomials, rational, trigonometric, exponential, logarithmic functions are continuous on their domain.
Examples

Find the limit
\[ \lim_{(x, y) \to (0, 0)} \frac{2x^2 - 4y^2}{\sqrt{2x^2 - 4y^2}} - 1 \]

Find the largest set on which the function
\[ 2xy + 9 - x^2 - y^2 \]
is continuous.
Find the limit

\[
\lim_{(x,y) \to (0,0)} \frac{2x^2 - 4y}{\sqrt{2x^2 - 4y + 1} - 1}
\]
Examples

Find the limit

\[
\lim_{{(x,y) \to (0,0)}} \frac{2x^2 - 4y}{\sqrt{2x^2 - 4y + 1} - 1}
\]

Find the largest set on which the function

\[
\frac{2xy}{9 - x^2 - y^2}
\]

is continuous.