1. Evaluate $\int x \cos^2 3x \, dx$

**Solution:** First rewrite $\cos^2 3x$ using the half-angle formula:

$$\int x \cos^2 3x \, dx = \int x \left( \frac{1 + \cos 6x}{2} \right) \, dx = \frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 6x \, dx.$$  

Now use integration by parts to evaluate $\int x \cos 6x \, dx$, setting $u = x$ and $dv = \cos 6x \, dx$, which makes $du = dx$ and $v = \frac{\sin 6x}{6}$:

$$\frac{1}{2} \int x \, dx + \frac{1}{2} \int x \cos 6x \, dx = \frac{x^2}{4} + \frac{x \sin 6x}{12} - \int \frac{\sin 6x}{12} \, dx$$

$$= \frac{x^2}{4} + \frac{x \sin 6x}{12} + \frac{\cos 6x}{72} + C.$$

---

2. Evaluate $\int e^{2x} \sin x \, dx$.

**Solution:** We use integration by parts twice. Set $I = \int e^{2x} \sin x \, dx$. Now, using integration by parts with $u = e^{2x}$ and $dv = \sin x \, dx$ (the other choice of $u$ and $dv$ works just as well), so $du = 2e^{2x} \, dx$ and $v = -\cos x$, we have

$$I = -e^{2x} \cos x - \int -2e^{2x} \cos x \, dx.$$  

Using integration by parts again with $u = -2e^{2x}$ and $dv = \cos x \, dx$, we get

$$I = -e^{2x} \cos x - \left( -2e^{2x} \sin x - \int -4e^{2x} \sin x \, dx \right)$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4 \int e^{2x} \sin x \, dx + C$$

$$= -e^{2x} \cos x + 2e^{2x} \sin x - 4I + C.$$  

Solving this equation for $I$, we see that

$$I = \frac{-e^{2x} \cos x + 2e^{2x} \sin x}{5} + C.$$
3. Could you in principle compute \( \int x^{10} e^x \, dx \), and if so, how?

**Solution:** Yes, using integration by parts 10 times, each time setting \( u \) equal to the polynomial and letting \( dv = e^x \, dx \).

4. Evaluate \( \int \sin^3 x \cos^4 x \, dx \).

**Solution:** Since the power of sine is odd, we convert 2 of the sines into cosines using \( \sin^2 x + \cos^2 x = 1 \), so \( \sin^2 x = 1 - \cos^2 x \):

\[
\int \sin^3 x \cos^4 x \, dx = \int \sin x (1 - \cos^2 x) \cos^4 x \, dx.
\]

Now we make a \( u \)-substitution, setting \( u = \cos x \) and \( du = -\sin x \, dx \):

\[
\int \sin x (1 - \cos^2 x) \cos^4 x \, dx = -\int (1 - u^2)u^4 \, du = -\int u^4 - u^6 \, du = -\frac{u^5}{5} + \frac{u^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C.
\]

5. Evaluate \( \int \sec^4 x \tan^4 x \, dx \).

**Solution:** Since the power of secant is even, we save a \( \sec^2 x \) and convert the other secants to tangents using the identity \( \sec^2 x = 1 + \tan^2 x \):

\[
\int \sec^4 x \tan^4 x \, dx = \int \sec^2 x (1 + \tan^2 x) \tan^4 x \, dx.
\]

Now we make a \( u \)-substitution, setting \( u = \tan x \) and \( du = \sec^2 x \, dx \):

\[
\int \sec^2 x (1 + \tan^2 x) \tan^4 x \, dx = \int (1 + u^2)u^4 \, du = \int u^4 + u^6 \, du = \frac{u^5}{5} + \frac{u^7}{7} + C = \frac{\tan^5 x}{5} + \frac{\tan^7 x}{7} + C.
\]
What substitution would you use to evaluate $\int x^3 \sqrt{16 + x^2} \, dx$?

Solution: We would like to simplify the radical $\sqrt{16 + x^2}$ using the identity $16 + 16 \tan^2 \theta = 16 \sec^2 \theta$, so we would set $x = 4 \tan \theta$.

Evaluate $\int \frac{dx}{(9 - x^2)^{3/2}}$.

Solution: The goal is to simplify $(9 - x^2)^{3/2}$ using the identity $9 - 9 \sin^2 \theta = 9 \cos^2 \theta$, so we set $x = 3 \sin \theta$, giving $dx = 3 \cos \theta$:

$$\int \frac{dx}{(9 - x^2)^{3/2}} = \int \frac{3 \cos \theta}{(9 - 9 \sin^2 \theta)^{3/2}} \, d\theta = \int \frac{3 \cos \theta}{3^3 \cos^3 \theta} \, d\theta = \frac{1}{9} \int \frac{d\theta}{\cos^2 \theta} = \frac{1}{9} \int \sec^2 \theta \, d\theta = \frac{1}{9} \tan \theta + C.$$

Now we would draw a right triangle with $\sin \theta = x/3$ to compute that $\tan \theta = x/\sqrt{9 - x^2}$, giving us that

$$\int \frac{dx}{(9 - x^2)^{3/2}} = \frac{x}{9\sqrt{9 - x^2}} + C.$$

Is the angle between the vectors $a = \langle 3, -1, 2 \rangle$ and $b = \langle 2, 2, 4 \rangle$ acute, obtuse, or right?

Solution: Since $a \cdot b = 6 - 2 + 8 = 12 > 0$ and $a \cdot b = |a||b| \cos \theta$, we see that $\cos \theta > 0$, so the angle between $a$ and $b$ is acute.
9. Find the area of the parallelogram whose vertices are \((-1, 2, 0), (0, 4, 2), (2, 1, -2),\) and \((3, 3, 0)\).

**Solution:**  Label the points \(P, Q, R,\) and \(S\). Then \(\overrightarrow{PQ} = (1, 2, 2), \overrightarrow{PR} = (3, -1, -2)\) and \(\overrightarrow{PS} = (4, 1, 0)\). It follows that the parallelogram is determined by \(\overrightarrow{PQ}\) and \(\overrightarrow{PR}\), so its area is \(|\overrightarrow{PQ} \times \overrightarrow{PR}| = |<-2, 8, -7>| = \sqrt{4 + 64 + 49} = \sqrt{117}\).

10. If \(a\) and \(b\) are both nonzero vectors and \(a \cdot b = |a \times b|\), what can you say about the relationship between \(a\) and \(b\)?

**Solution:**  We are given that \(a \cdot b = |a \times b|\), and we know that

\[
\begin{align*}
\mathbf{a} \cdot \mathbf{b} &= |\mathbf{a}||\mathbf{b}| \cos \theta, \text{ while} \\
|\mathbf{a} \times \mathbf{b}| &= |\mathbf{a}||\mathbf{b}| \sin \theta
\end{align*}
\]

It follows that we must have \(\cos \theta = \sin \theta\), and the only value of \(\theta\) which satisfies this is \(\theta = \pi/4\), so the two vectors are at a 45° angle to each other.

11. Consider the vectors \(a = (4, 1)\) and \(b = (2, 2)\), shown below. Compute \(\cos \theta, u,\) and the length \(x\).

![Diagram of vectors](image)

**Solution:**  As \(\theta\) is the angle between \(a\) and \(b\), we can find it via the dot product:

\[
\mathbf{a} \cdot \mathbf{b} = 10 = |\mathbf{a}||\mathbf{b}| \cos \theta = \sqrt{17}\sqrt{8} \cos \theta,
\]

so \(\cos \theta = \frac{10}{\sqrt{17}\sqrt{8}}\).

Now,

\[
\mathbf{u} = \text{proj}_a \mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \mathbf{a} = \frac{10}{17} (4, 1) = \left\langle \frac{40}{17}, \frac{10}{17} \right\rangle.
\]

Finally, using the Pythagorean Theorem,

\[
x = \sqrt{|\mathbf{b}|^2 - |\mathbf{u}|^2} = \sqrt{8 - \frac{100}{17}} = \sqrt{\frac{36}{17}} = \frac{6}{\sqrt{16}}.
\]
12 Find the equation of the plane which passes through the point \((2, -3, -1)\) and contains the line
\[
x = 3t - 2, \quad y = t + 3, \quad z = 5t - 3.
\]

**Solution:** We need to find two vectors on this plane, so consider the vector from the point \((2, -3, -1)\) to the point \((-2, 3, -3)\) which lies on the given line (just set \(t = 0\) in the line equation). This vector is \((-4, 6, 2)\), and the direction vector of the given line is \((3, 1, 5)\), so the normal vector to the plane is \((-4, 6, 2) \times (3, 1, 5) = (28, 26, -22)\). The equation for the plane is then
\[
28(x - 2) + 26(y + 3) - 22(z + 1) = 0.
\]

13 Find the line of intersection of the planes \(x + y + z = 12\) and \(2x + 3y + z = 2\).

**Solution:** Since the line of intersection lies on both planes, it must be orthogonal to both normal vectors. Therefore its direction is given by the cross product of the normal vectors:
\[
(1, 1, 1) \times (2, 3, 1) = (-2, 1, 1).
\]

We also need a point on the line of intersection. To find this, let us set \(x = 0\) (other choices work equally well). The equations for the first plane becomes \(y + z = 12\), so \(z = 12 - y\). Substituting this into the equation for the second plane gives \(3y + (12 - y) = 2\), so \(y = -5\). Thus the point \((0, -5, 17)\) lies on the line of intersection, so the line is given by
\[
x = -2t, \quad y = t - 5, \quad z = t + 17.
\]

14 Compute the position vector for a particle which passes through the origin at time \(t = 0\) and has velocity vector
\[
v(t) = 2t \mathbf{i} + \sin t \mathbf{j} + \cos t \mathbf{k}.
\]

**Solution:** The position vector is the antiderivative of the velocity vector, so it is
\[
r(t) = \int v(t) \, dt = t^2 \mathbf{i} - \cos t \mathbf{j} + \sin t \mathbf{k} + \mathbf{C},
\]
where $C$ is a vector constant of integration. The problem stated that the particle passes through origin at time $t = 0$, so need $r(0) = 0$:

$$0 = r(0) = -j + C;$$

thus $C = j$, so we have that

$$r(t) t^2 i + (1 - \cos t) j - \sin t k.$$ 

15. Show that if a particle moves at constant speed, then its velocity and acceleration vectors are orthogonal. Note that this does not mean that the velocity is 0! (Hint: consider the derivative of $v \cdot v$.)

**Solution:** Suppose that the particle’s speed is $C$, so $|v(t)| = C$. Then we have

$$v(t) \cdot v(t) = C^2,$$

so taking the derivatives of both sides gives

$$v(t) \cdot v'(t) + v'(t) \cdot v(t) = 0,$$

which implies that $v(t) \cdot v'(t) = v(t) \cdot a(t) = 0$, as we wanted.

16. Consider the curve defined by

$$r(t) = \langle 4 \sin ct, 3ct, 4 \cos ct \rangle .$$

What value of $c$ makes the arc length of the space curve traced by $r(t)$, $0 \leq t \leq 1$, equal to 10?

**Solution:** The arc length from 0 to 1 of this curve is given by

$$\int_0^1 \text{speed} \, dt = \int_0^1 \sqrt{16c^2 \cos^2 ct + 9c^2 + 16c^2 \sin^2 ct} \, dt$$

$$= \int_0^1 \sqrt{16c^2 + 9c^2} \, dt$$

$$= \int_0^1 5c \, dt$$

$$= 5c.$$

For this to equal 10, we want $c = 2$. 