Crossings and patterns in signed permutations

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Permutation Patterns
A crossing of a permutation $\sigma$ is a couple $(i, j)$ such that $i < j \leq \sigma(i) < \sigma(j)$, or $\sigma(i) < \sigma(j) < i < j$.

The crossings and 13-2 are equidistributed in permutations. This is also the same as superfluous ones in permutation tableaux. [Corteel Nadeau, Steingrímsson Williams]
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Introduction

Definition
An occurrence of the pattern 13-2 in \( \sigma \in \mathcal{S}_n \) is a triple \((i, i + 1, j)\) such that \(\sigma(i) < \sigma(j) < \sigma(i + 1)\).

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Example

```
1 2 3 4 5
```

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5 / 28
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Example
Definition

A permutation tableau is a Young diagram filled with 0’s and 1’s, such that:

- There is at least a 1 per column,

\[
\begin{array}{cccc}
1 & & & 1 \\
& 1 & 0 & 1 \\
& 0 & 0 & 1 \\
1 & 1 & & \\
\end{array}
\]

- The pattern is forbidden.

\[
\begin{array}{cccc}
1 & \ldots & 0 \\
1 & & & \\
& 1 & 1 & \\
\end{array}
\]
Type $B$ permutation tableaux: defined by Lam and Williams (in relation with geometric objects such as orthogonal grassmannian...)

These are roughly conjugate-symmetric permutation tableaux, and are in bijection with signed permutations.

Question: are there some notions of crossings and patterns for signed permutations?
Type B permutation tableaux

Remark: A conjugate-symmetric permutation tableau contains no zero-row.

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```
1 0 1 0 0
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1 1 1 0
0 0 0
0 1
```

OK
The zig-zag bijection

We use a bijection of [Steingrímsson Williams]. Label the boundary of the permutation tableau with integers for $-n$ to $n$. The image of $i$ is obtained by taking a zig-zag path, the direction East or South changing at each 1.

Example

\[
\begin{array}{cccc}
1 & 0 & 1 & 1 \\
0 & 0 & 1 & -1 \\
1 & 1 & 1 & -1 \\
1 & 3 & 2 & -4 \\
\end{array}
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$\pi = 3, 1, 4, -2$. 

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Crossings for signed permutations

**Definition**
A *crossing* of a signed permutation is a pair \((i, j) \in [n]^2\) such that
- either \(i < j \leq \pi(i) < \pi(j)\),
- or \(i > j > \pi(i) > \pi(j)\),
- or \(-i < j \leq -\pi(i) < \pi(j)\).

We use an arrow notation such that this corresponds to proper intersection between arrows, or the limit case of two arrows.

**Example**
\[
\pi = 3, 1, 4, -2.
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Theorem
Via the zig-zag bijection,

- the number of superfluous 1's in type B permutation tableaux is the number of crossings in signed permutations,
- \( i > 0 \) is such that \( \pi(i) \geq i \) iff \( i \) label a vertical step in the South-East boundary of the permutation tableau,
- the number of \( i > 0 \) with \( \pi(i) < 0 \) is the number of 1's in the diagonal of the permutation tableau.

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Eulerian numbers of type $B$

There are some definitions of ascents and exceedances in signed permutations [Brenti, Chow] whose distribution are type $B$ Eulerian numbers. In our context, it is interesting to define:

$$\text{twex}(\pi) = \#\{ i | \pi(i) \geq i \} + \left\lfloor \frac{\text{neg}(\pi)}{2} \right\rfloor$$

**Theorem**

Let

$$B_{n,k}(q) = \sum_{\pi \text{ with } \text{twex}(\pi)=k} q^{\text{cr}(\pi)},$$

Then $B_{n,k}(q)$ is a $q$-analog of type $B$ Eulerian numbers such that $B_{n,k}(q) = B_{n,n-k}(q)$. 
Non-crossing partitions

A set partition is non-crossing if there are no $i < j < k < \ell$ with $i, j$ in a same block, $k, \ell$ a one other block.

There is a bijection between non-crossing permutations and non-crossing partitions given by the cycle decomposition.

$$\pi = \left\{\{1, 4, 8\}, \{2, 3\}, \{5, 6, 7\}\right\}$$

Similarly, non-crossing signed permutations are in bijection with non-crossing partitions of type $B$. 
Non-crossing partition of classical types are defined as a sublattice of a Coxeter group. Combinatorial description in type $B$: a type $B$ non-crossing partition is a couple a (type $A$) non-crossing partition, and a subset of the non-nested blocks.

There is a bijection with signed permutations having no crossing, for example with $\pi = -2, 1, -7, 3, 6, 5, 4$:

![Diagram of non-crossing partitions]

We have $B_{n,k}(0) = \binom{n}{k}^2$, the Narayana number of type $B$. 
A pattern for signed permutation?

There is a definition of “31-2” pattern for signed permutation such that the distribution is the same as crossings, and an associated notion of signed ascents such that 31-2 gives a $q$-analog of type $B$ Eulerian numbers.

**Definition**

- $31-2(\pi) = \# \{ (i, j) \mid \text{such that } i < j, \text{and } |\pi(i)| > |\pi(j)| > |\pi(i + 1)| \text{ or } \pi(i) > -\pi(j) \geq |\pi(i + 1)| \}$
- $\text{pasc}(\pi) = \# \{ i \mid \}$

The proof is quite indirect: there is a recursive decomposition of type $B$ permutation tableaux that can be interpreted in terms of weighted Motzkin paths, and then there is a bijection between paths and signed permutations.
Conclusion

- Are there nice enumeration formulas for crossings in signed permutations (case of permutations: [J-V, Corteel, Rubey, Prellberg])?
- Is there a better definition of the signed pattern 31-2?
- Snakes defined by Arnol’d are the signed analog of alternating permutations, are our statistic useful in this context?
thanks
for your
attention