Permutation Patterns in Algebraic Geometry

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Table of Contents

1 Definitions
   • ... from Combinatorics
   • ... from Geometry

2 Patterns determine geometry
   • Three geometric properties of varieties
   • Description in terms of patterns

3 Open problems
We start with some definitions.
Permutations and patterns

A permutation in $\mathcal{S}_n$ is a bijection $\pi: \{1, \ldots, n\} \rightarrow \{1, \ldots, n\}$. We will use one-line notation for permutations, for example, $\pi = 32415$ is the permutation in $\mathcal{S}_5$ that sends

1 $\mapsto$ 3
2 $\mapsto$ 2
3 $\mapsto$ 4
4 $\mapsto$ 1
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Patterns are also permutations but we are interested in how they occur in other permutations . . .
Patterns inside permutations

Given a pattern $p$ we say that it **occurs** in a permutation $\pi$ if $\pi$ contains a subsequence that is order-equivalent to $p$. If $p$ does not occur in $\pi$ we say that $\pi$ **avoids** the pattern $p$. Let $\mathcal{S}_n(p)$ denote the set of permutations in $\mathcal{S}_n$ that avoid the pattern $p$.

**Example**

The permutation $\pi = 32415$ has two occurrences of the pattern

\[
123 = \begin{array}{c}
\bullet \\
\end{array} \begin{array}{c}
\bullet \\
\end{array} : 32415, \quad 32415
\]

It avoids the pattern

\[
132 = \begin{array}{c}
\bullet \\
\end{array} \begin{array}{c}
\bullet \\
\end{array} .
\]
Babson and Steingrímsson (2000) defined generalized patterns, or vincular patterns, where conditions are placed on the locations of the occurrence.

Example

The permutation $\pi = 32415$ has one occurrence of the pattern

$$123 = \begin{array}{ccc}
\cdot & & \\
\cdot & & \\
\cdot & \cdot & \\
\end{array} : 32415$$

It avoids the pattern

$$123 = \begin{array}{cc}
\cdot & \\
\cdot & \\
\cdot & \cdot & \\
\end{array}.$$
Motivation for vincular patterns

- They simplify descriptions given in terms of more complicated patterns – we’ll see this later when we look at factorial Schubert varieties.
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- Many interesting sequences of integers come up when we count the permutations avoiding a pattern $p$. For example if $p$ is any classical pattern of length 3 then

$$|\mathcal{S}_n(p)| = \text{n-th Catalan number} = \frac{1}{n+1} \binom{2n}{n}.$$ 

However Claesson showed in 2001 that

$$|\mathcal{S}_n(123)| = \text{n-th Bell number}.$$
Bivincular patterns

Bousquet-Mélou, Claesson, Dukes, and Kitaev (2010) defined **bivincular patterns** as vincular patterns with extra restrictions on the values in an occurrence.

**Example**

The permutation $\pi = 32415$ has one occurrence of the pattern

\[
\begin{array}{c}
123 \\
\overline{123}
\end{array} =
\begin{array}{c}
\ \ \ \\
\bullet \\
\bullet
\end{array} : 
32415
\]

This is not an occurrence of $\overline{123}$. But it is an occurrence of

\[
\begin{array}{c}
123 \\
\overline{123}
\end{array} =
\begin{array}{c}
\ \ \ \\
\bullet \\
\bullet
\end{array} : 
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- New Wilf-equivalence: For example the patterns

12345678
76128543

are equivalent.
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- They simplify descriptions given in terms of more complicated patterns – we’ll see this later when we look at Gorenstein Schubert varieties.
- More interesting sequences of integers.
- New Wilf-equivalence: For example the patterns

\[
\begin{align*}
&1 \\ &2 \\ &3 \\ &4 \\ &5 \\ &6 \\ &7 \\ &8
\end{align*}
\]

\[
\begin{align*}
&12345678 \\
&65418723
\end{align*}
\]

are equivalent.
(Complete) flags

We will only consider complete flags in $\mathbb{C}^m$ so we will simply refer to them as flags. A flag is a sequence of vector-subspaces of $\mathbb{C}^m$

$$E_\bullet = (E_1 \subset E_2 \subset \cdots \subset E_m = \mathbb{C}^m),$$

with the property that $\dim E_i = i$. The set of all such flags is called the (complete) flag manifold, and denoted by $F_\ell(\mathbb{C}^m)$. 
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with the property that $\dim E_i = i$. The set of all such flags is called the (complete) flag manifold, and denoted by $F\ell(\mathbb{C}^m)$. We want to consider special subsets of this flag manifold ...
Schubert cells in $F\ell(\mathbb{C}^m)$

If we choose a basis $f_1, f_2, \ldots, f_m$, for $\mathbb{C}^m$ then we can fix a reference flag

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such that $F_i$ is spanned by the first $i$ basis vectors.
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$$\dim(E_p \cap F_q) = \#\{i \leq p \mid \pi(i) \leq q\},$$

for $1 \leq p, q \leq m$. 
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$$\dim(E_p \cap F_q) = \#\{i \leq p \mid \pi(i) \leq q\},$$

for $1 \leq p, q \leq m$. Let’s look at an example.
A Schubert cell in $F_{\ell}(\mathbb{C}^3)$

Let $\pi = 231$. The conditions for the Schubert cell $X_{231}^\circ$

$$\dim(E_p \cap F_q) = \# \{ i \leq p \mid \pi(i) \leq q \},$$

become
A Schubert cell in $F_\ell(\mathbb{C}^3)$

Let $\pi = 231$. The conditions for the Schubert cell $X_{231}^0$

$$\dim(E_p \cap F_q) = \# \{ i \leq p \mid \pi(i) \leq q \},$$

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$E_1, E_2$ intersect $F_1$ in a point
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$E_1$, $E_2$ intersect $F_1$ in a point

$E_1 \subset F_2$, $E_2 \cap F_2 = E_1$
Schubert varieties in $F\ell(\mathbb{C}^m)$

Given a Schubert cell $X^\circ_\pi$ we define the **Schubert variety** as the closure

$$X_\pi = \overline{X^\circ_\pi},$$

in the Zariski topology.
We will now show how pattern avoidance can be used to describe geometric properties of Schubert varieties.
Smooth, factorial and Gorenstein varieties

Pictorial definition of smoothness: the tangent space at every point has the right dimension.

(a) $y - x^2 = 0$.
(b) $y^2 - x^2 - x^3 = 0$.

Figure: Compare the single tangent direction in subfigure 1(a) with the two tangent directions in subfigure 1(b).
Smooth, factorial and Gorenstein varieties

Algebraic definitions: a variety:

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Three geometric properties of varieties

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\[
\begin{array}{|c|c|}
\hline
X_\pi \text{ is} & \text{if} \\
\hline
\text{smooth} & \pi \text{ avoids } 2143 \text{ and } 1324 \\
\hline
\text{factorial} & \pi \text{ avoids } 21354 \text{ and } 1324 \\
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\text{Gorenstein} & \text{the local rings are Gorenstein} \\
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Gorenstein Schubert varieties in terms of bivincular patterns

The short answer to the question is “yes”. The long answer should include that it is much more complicated than I had originally hoped.
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- The first condition of factoriality, avoiding $2143$, is weakened to

  avoiding $\begin{array}{c} 12345 \\ 31524 \end{array}$ and $\begin{array}{c} 12345 \\ 24153 \end{array}$. 

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- The first condition of factoriality, avoiding $2143$, is weakened to

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- The second condition of factoriality, avoiding $1324$, is weakened to the avoidance of two infinite families of bivincular patterns, which we now describe.
The associated partition of a permutation

Here we will only consider permutations with a unique descent, as this allows us to avoid a minor technical detail.
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Example

The permutation

\[ \pi = 134892567|10 \]

has a unique descent at \( d = 5 \).
Example

The permutation

$$\pi = 13489 \downarrow 2567|10$$

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Figure: A bounding box with dimensions $5 \times 5$. 
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Figure: Outer corners of $\pi = 13489 \downarrow 2567|10$. 
Depth and width of outer corners

We see that all the inner corners lie on the same diagonal if and only each outer corner has the same **depth** and **width**.

Figure: $\pi = 13589 \downarrow 2467|10$. 
Detecting too wide outer corners

Let’s go back to the permutation \( \pi = 13489 \downarrow 2567|10 \), and consider the outer corner that is too wide.
Definitions

Patterns determine geometry

Open problems

Description in terms of patterns

Detecting too wide outer corners

Let’s go back to the permutation \( \pi = 13489 \downarrow 2567|10 \), and consider the outer corner that is too wide.

This outer corner comes from the subsequence \( 13489 \downarrow 2567|10 \).
Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern

\[ \begin{array}{c}
1234567 \\
1562347
\end{array} = \begin{array}{c}
\cdot \\
\cdot \\
\end{array} \]
Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
15 & 6 & 2 & 3 & 4 & 7
\end{bmatrix}
\]

In general, we can detect too wide outer corners with the patterns

\[
\begin{align*}
&\overline{12345} \quad \overline{1234567} \quad \overline{123456789} \quad \overline{12\cdot\cdots\cdot\cdots\cdot k} \\
&14235 \quad 1562347 \quad 167823459 \quad \cdots \quad 1\ell+1 \cdot 2 \cdot \ell k \quad \cdots
\end{align*}
\]
Detecting too wide outer corners cont.

The shape of this outer corner can be detected with the bivincular pattern

\[
\begin{array}{c}
1234567 \\
1562347
\end{array}
\]

In general, we can detect too wide outer corners with the patterns

\[
\begin{array}{cccccc}
12345 & 1234567 & 123456789 & 12\cdots k \\
14235 & 1562347 & 167823459 & 1\ell+1 \cdots 2 \cdots \ell k
\end{array}
\]

and too deep outer corners with the patterns

\[
\begin{array}{cccccc}
12345 & 1234567 & 123456789 & 12\cdots k \\
13425 & 1456237 & 156782349 & 1\ell+1 \cdots 2 \cdots \ell k
\end{array}
\]
The Schubert variety

<table>
<thead>
<tr>
<th>( X_\pi ) is</th>
<th>if</th>
</tr>
</thead>
<tbody>
<tr>
<td>smooth</td>
<td>( \pi ) avoids 2143 and 1324</td>
</tr>
<tr>
<td>factorial</td>
<td>( \pi ) avoids 21_43_ and 1324</td>
</tr>
<tr>
<td>Gorenstein</td>
<td>( \pi ) avoids ( \frac{12345}{31524} ) and ( \frac{12345}{24153} ), ...</td>
</tr>
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...and the two infinite corner families — remember that this is modulo a technical detail I have omitted.
The Schubert variety

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... and the two infinite corner families — remember that this is modulo a technical detail I have omitted.
The description is in terms of patterns only and one doesn’t need to construct the associated partition.
Benefits from the bivincular description

- The description is in terms of patterns only and one doesn’t need to construct the associated partition.
- It is very easy to see on the pattern level that smooth implies factorial implies Gorenstein.
We end with some open problems.
A variety is a **local complete intersection** if it can be described by the expected number of equations. This condition is in between factoriality and Gorensteinness and I’m working with Woo on giving a pattern description.
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Recall that weakening smoothness to factoriality meant adding an underline in one of the patterns. It would be interesting to know what geometric property is described by the addition of more underlines and overlines.
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Where do the **mesh patterns** patterns fit into this story?
The end! Questions?