Written Certification Exam, Day 1
June 18, 2012, 9:00am – 12:00pm

(1) Let $p : X' \to X$ be a covering map, and assume that $X'$ is path-connected. Let $x_0, x_1 \in X'$ and $x \in X$ be points such that $p(x_0) = x = p(x_1)$. Prove that the subgroups $p_*\pi_1(X',x_0)$ and $p_*\pi_1(X',x_1)$ are conjugate in $\pi_1(X,x)$.

(2) Ideals and quotients.
(a) Find all ideals of the quotient ring $\mathbb{Q}[x]/(x^{14} - 1)$. In particular, how many such ideals are there?
(b) Determine the structure of the quotient ring $\mathbb{Z}[x]/(5, x^2 - 2)$. Be as precise as you can.

(3) Of the following smooth manifolds, which ones admit a continuous nowhere vanishing vector field:
- $S^2$ minus a point.
- $S^2$
- $S^3$
- $S^1 \times S^1$
- $SL(n, \mathbb{R})$
- An oriented compact surface of genus three with no boundary.

(4) Let $f : A \subset \mathbb{R} \to \mathbb{R}$ be a function. Give three criteria ($\epsilon/\delta$, open sets, sequences) for $f$ to be continuous on $A$. Show that these definitions are equivalent.

(5) Let $\Omega$ be an open connected subset of $\mathbb{C}$. Suppose that $f_n$ is holomorphic$^1$ on $\Omega$ for each $n \geq 1$ and that the sequence $\{f_n\}$ converges to a function $f$ uniformly on each compact subset of $\Omega$.
(a) Show that $f$ is holomorphic on $\Omega$.
(b) Show that the sequence $\{f'_n\}$ of derivatives converges to $f'$ uniformly on compact subsets of $\Omega$.

(6) Let $L$ be the splitting field over $\mathbb{Q}$ of $x^9 - 8$.
(a) Determine the degree $[L : \mathbb{Q}]$ carefully explaining all conclusions.
(b) Justify whether or not the Galois group $\text{Gal}(L/\mathbb{Q})$ is abelian.
(c) Justify whether or not the Galois group $\text{Gal}(L/\mathbb{Q})$ is solvable.

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$^1$We say that $g$ is holomorphic on $\Omega$ if $g'(z)$ exists for all $z \in \Omega$. 
Written Qualification Exam, Day 2

June 19, 2012, 9:00am – 12:00pm

(1) Let the field \( K \) be an extension field of a field \( k \). Show that there is a natural isomorphism of \( K \)-algebras \( K \otimes_k M_n(k) \rightarrow M_n(K) \), where for a ring \( R \), \( M_n(R) \) denotes the ring of \( n \times n \) matrices over \( R \).

(2) Denote by \( S^n \) the unit sphere in \( \mathbb{R}^{n+1} \). If \( F : S^n \rightarrow S^n \) is the antipodal map defined by \( F(x) = -x \), then show by calculation, that the degree of \( F \) is \((-1)^{n+1}\).

(3) Let \( C([0,1]) \) be the complex vector space of continuous complex-valued functions on \([0,1]\).

(a) Suppose that \( \{f_n\} \) is a sequence in \( C([0,1]) \) and that \( f \) is a function on \([0,1]\) such that \( f_n \) converges uniformly to \( f \). Show that \( f \in C([0,1]) \).

(b) Assume without proof that
\[
\|f\|_{\infty} := \sup \{ |f(t)| : t \in [0,1] \}
\]
is a norm on \( C([0,1]) \). Show that \( C([0,1]) \) is a Banach space with respect to \( \|\cdot\|_{\infty} \).

(4) Let \( T \) be a linear operator on a finite dimensional vector space \( V \) defined over a field \( k \). Let \( \chi_T(x) = (x - \lambda_1)^{m_1} \cdots (x - \lambda_r)^{m_r} \) be the characteristic polynomial, and assume all the \( \lambda_i \) are distinct. Let \( V_i \) be the eigenspace corresponding to the eigenvalue \( \lambda_i \).

(a) Show that \( \dim V_i \geq 1 \) for all \( i, 1 \leq i \leq r \).

(b) Choose nonzero \( v_i \in V_i \). Show that \( \{v_1, \ldots, v_r\} \) is linearly independent.

(c) Conclude that if \( \dim V_i = m_i \) for all \( i \), then \( T \) is diagonalizable.

(5) Determine the singular homology groups of the standard torus (i.e., regarded as an identification space of a 2-dimensional rectangle) using the Mayer-Vietoris sequence.

(6) Let \( \mathcal{H} \) be a complex Hilbert space and \( T : \mathcal{H} \rightarrow \mathcal{H} \) a linear map.

(a) Show that if \( T \) is bounded, then there is a linear map \( S : \mathcal{H} \rightarrow \mathcal{H} \) such that \((Tv \mid w) = (v \mid Sw)\) for all \( v, w \in \mathcal{H} \). (In other words, show that \( T \) has an adjoint.)

(b) Conversely, show that if there is a (not necessarily bounded) map \( S : \mathcal{H} \rightarrow \mathcal{H} \) such that \((Tv \mid w) = (v \mid Sw)\) for all \( v, w \in \mathcal{H} \), then \( T \) is bounded.
(1) Show that any group of order 30 is the semidirect product of two smaller abelian groups.

(2) Let \( f \) be a complex function on an open connected subset \( \Omega \) of the complex plane.
(a) What are the Cauchy-Riemann equations for \( f \) at \( z_0 \in \Omega \)?
(b) Discuss the existence of the complex derivative \( f'(z_0) \) in terms of the Cauchy-Riemann equations at \( z_0 \). (Ideally, you should give both necessary as well as sufficient conditions for \( f'(z_0) \) to exist. Note that you are not asked to prove anything here.)
(c) Show that a real-valued function on \( \Omega \) is holomorphic if and only if it is constant.

(3) Let \( m > 1 \) be a square-free integer, and \( n \geq 1 \) an odd integer. Let \( \mathbb{F}/\mathbb{Q} \) be any field extension with \([\mathbb{F} : \mathbb{Q}] = 2\). Show that \( x^n - m \) is irreducible in the polynomial ring \( \mathbb{F}[x] \).

(4) Let \( \mathcal{H} \) be a complex Hilbert space and \( T : \mathcal{H} \to \mathcal{H} \) a linear map.
(a) What does it mean for \( T \) to be bounded?
(b) Define the operator norm, \( ||T|| \), of \( T \) and show that \( ||Th|| \leq ||T|| \cdot ||h|| \) for all \( h \in \mathcal{H} \).
(c) Show that \( T \) is bounded if and only if \( T \) is continuous from \( \mathcal{H} \) to \( \mathcal{H} \).

(5) Let \( \phi_1 \) and \( \phi_2 \) be two charts on \( \mathbb{R} \) defined by \( \phi_1(t) = t \) and \( \phi_2(t) = t^3 \). Are they \( C^\infty \) compatible? Prove your answer.

(6) Define the wedge product of two differential forms on a manifold. How does one use this operation to define the cup product of two de Rham cohomology classes? Prove that the cup product is well defined.