Algebra questions

1. Let $L$ be the splitting field of $x^{15} - 8$ over $\mathbb{Q}$, and let $G$ be the Galois group $\text{Gal}(L/\mathbb{Q})$.

   (a) Show that $G$ is a semidirect product of two proper subgroups $K$ and $H$.

   (b) Identify the subgroups $K, H$ as subgroups of the Galois group, i.e., in terms of intermediate fields, and determine their isomorphism types.

2. Give three equivalent conditions which characterize when an algebraic extension of fields $L/K$ is a normal extension, and prove any two are equivalent.

3. Let $F$ be a field of characteristic 0, $f \in F[x]$ an irreducible polynomial of degree $n \geq 1$, and $K$ the splitting field of $f$ over $F$. It should be well-known that $[K : F] \leq n!$. The point of this problem is to show $[K : F] \mid n!$. Hint: prove that there exists an injective homomorphism $\text{Gal}(K/F) \to S_n$ where $S_n$ is the symmetric group on $n$ letters.

4. Let $G$ be a group of order $pqr$, where $p < q < r$ are distinct primes. Show that $G$ is solvable.

5. Let $K$ be the subgroup of $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ generated by the three elements: $u_1 = (1, -3, -2)$, $u_2 = (1, 3, 2)$, and $u_3 = (3, 3, 4)$. Determine the structure of the quotient $G/K$ as a direct sum of cyclic groups.

6. Let $R$ be a commutative ring. An $R$-module $M$ is flat if the functor $M \otimes_R (\cdot)$ is exact. Prove that any projective $R$-module is flat.

Topology questions

1. Let $M$ be a smooth manifold, and let $x^1, \ldots, x^n$ be a local coordinate system defined on an open set $U \subseteq M$. Consider the $(1, 1)$-tensor field $C$ defined on $U$ in local coordinates by

   $$C = \sum_{i=1}^{n} dx^i \otimes \frac{\partial}{\partial x^i}.$$ 

   Show that $C$ is independent of the choice of local coordinates and hence defines a smooth global tensor field on $M$.

2. Determine the the critical points of the determinant mapping $\det: M_n(\mathbb{R}) \to \mathbb{R}$ defined on the space of $n \times n$ matrices. [Hint: The determinant is multilinear as a function of the columns of a matrix.]
3. Let $S \subseteq \mathbb{R}^3$ be the surface with boundary given by

$$S = \{(x, y, z) : z = x^2 + y^2, z \leq 9\},$$

oriented by the unit normal field $N = (n_1, n_2, n_3)$ with $n_3 < 0$. Let $\omega$ be the 2-form on $\mathbb{R}^3$ given by

$$\omega = e^{z \sin y} dy \wedge dz + \tan^{-1}(x \sinh z) dz \wedge dx + 2 dx \wedge dy.$$

Compute the integral $\int_S \omega$.

4. Suppose that a space $X$ is the disjoint union $X = U \sqcup V$ of two open subspaces $U$ and $V$.

(a) Use the Eilenberg-Steenrod axioms to prove that for any homology theory, the homology groups of $X$ are given in terms of those of $U$ and $V$ by

$$H_q(X) = H_q(U) \oplus H_q(V).$$

(b) Why is this result easier if we take the homology theory to be singular homology?

5. Let $p : Y \to X$ be a covering map. Let $Z$ be any connected space, and let $f : Z \to X$ be a continuous map. Suppose that $f_1 : Z \to Y$ and $f_2 : Z \to Y$ are continuous lifts of $f$ (i.e., $p \circ f_i = f$ for $i = 1, 2$) that agree at some point $z_0 \in Z$. Show that $f_1 = f_2$ on all of $Z$.

6. Consider the space $X$ obtained as the quotient space of a planar hexagon and its interior by identifying boundary edges of the hexagon in pairs according to the following scheme:

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1 ← a ← 2
c

6
b

3
c

5 ← a ← 4
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Compute the homology groups of $X$.

**Analysis questions**

1. State the Hahn-Banach Theorem and use it to show that if $B$ is a Banach space, then its dual, $B^*$, of bounded linear functionals separates points of $B$. (That is, you are asked to show that if $a$ and $b$ are distinct elements of $B$, then there is a $\phi \in B^*$ such that $\phi(a) \neq \phi(b)$.)
2. State the Residue Theorem (from Complex Analysis) and use it to evaluate
\[ \int_0^\infty \frac{x^2}{(x^2 + a^2)^2} \, dx \quad \text{for } a > 0. \]
Be sure to justify any limits required.

3. Consider a power series
\[ \sum_{n=1}^\infty a_n x^n \quad (\dagger) \]
for real constants \( a_n \in \mathbb{R} \). Show that there is a \( \rho \in [0, \infty] \) such that either
(i) \( \rho = 0 \) by which we mean (\dagger) converges only for \( x = 0 \), or
(ii) \( \rho = \infty \) by which we mean (\dagger) converges absolutely for all \( x \), or
(iii) \( 0 < \phi < \infty \) and (\dagger) converges absolutely if \( |x| < \rho \) and diverges if \( |x| > \rho \).
Give examples (with all \( a_n \neq 0 \)) where \( \rho = 0 \), \( \rho = \infty \) and \( 0 < \phi < \infty \).

4. Give a precise statement of the theorem which implies that a holomorphic function on an open subset of the complex plain is locally represented by a power series. Use your theorem to calculate the radius of convergence of the MacLaurin series for
\[ f(z) = \frac{1}{1 + e^z}. \]
(The MacLaurin series is just the Taylor series for \( f \) about \( z = 0 \).)

5. Let \( \mu \) be a measure on the Borel sets of \( \mathbb{R} \) such that for any Borel set \( E \subseteq \mathbb{R} \) we have
\[ \mu(E) = \inf \{ \mu(U) : U \text{ is an open set containing } E \} \]
and \( \mu([a, b]) < \infty \) for an interval \([a, b]\).

(i) Show that for any \( \epsilon > 0 \) there is an open set \( O \) and a closed set \( C \) such that \( C \subseteq E \subseteq O \) and \( \mu(O \setminus C) < \epsilon \).

(ii) Using the above, show that there are Borel sets \( G \) and \( F \) such that \( F \subseteq E \subseteq G \) with \( \mu(G \setminus F) = 0 \).

(Hint: Finding a neighborhood \( O \) of \( E \) such that \( \mu(O \setminus E) < \epsilon \) is pretty easy if \( \mu(E) < \infty \).)

6. Show that a continuous function \( f : (0, 1] \to \mathbb{R} \) is uniformly continuous if and only if there is continuous extension \( g : [0, 1] \to \mathbb{R} \). (That is, \( g \) is a continuous function such that \( g(x) = f(x) \) for all \( x \in (0, 1] \).)