1. Let $V$ be a 3-dimensional $\mathbb{Q}$-vector space, and let $T : V \to V$ be a linear operator that has eigenvalues 1 and 2 but is not diagonalizable.

   (a) What are the possible rational canonical forms of $T$?

   (b) What are the possible Jordan canonical forms of the operator $\text{Id} \otimes T : \mathbb{C} \otimes \mathbb{Q} V \to \mathbb{C} \otimes \mathbb{Q} V$ on the complexification?

2. Let $A$ be an integral domain.

   (a) Define what it means for an element $\pi \in A$ to be irreducible.

   (b) Suppose that $\pi \in A$ is irreducible. Show that the polynomial ring $A[x]$ is not a PID.

   (c) Show that $A[x]$ is a PID if and only if $A$ is a field.

3. Let $V$ be a finite-dimensional vector space over a field $k$ of characteristic zero, and let $\langle \cdot, \cdot \rangle : V \times V \to k$ be a skew-symmetric bilinear form.

   (a) State what it means to say that the form is nondegenerate.

   (b) Let $W \subseteq V$ be a subspace of such that the restriction $\langle \cdot, \cdot \rangle : |_{W \times W} : W \times W \to k$ is nondegenerate. Show that $V$ admits an orthogonal decomposition $V = W \oplus W^\perp$, where $W^\perp = \{ x \in V : \forall w \in W, \langle x, w \rangle = 0 \}$. Show also that if the bilinear form on $V$ was nondegenerate, then so is its restriction to $W^\perp$.

   (c) Show that if the form is nondegenerate on $V$, then $V$ is even-dimensional, and it has a basis relative to which the Gram matrix of the form is

   $\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$,

   where $I_n$ is the $n \times n$ identity matrix.

4. Let $K$ be a field of prime characteristic $p$, $\mathbb{F}_p$ the finite field with $p$ elements.

   (a) First assume that $K/\mathbb{F}_p$ is an algebraic extension. Show that for every $\alpha \in K$, there is a unique $\beta \in K$ with $\beta^p = \alpha$.

   (b) Now let $K$ be an arbitrary field of characteristic $p$, and assume that $L/K$ is a finite extension with $[L : K] = n$ and $p \nmid n$. Show that $L/K$ is a separable extension of fields.
5. A nonabelian group $G$ has exactly three conjugacy classes. What group is $G$, and why?

6. Let $n = 13 \cdot 29 = 377$, and $m \geq 3$ a square-free integer. Let $L$ be the splitting field over $\mathbb{Q}$ of $(x^7 - m)(x^n - 1)$.

   (a) Determine the splitting field $L/\mathbb{Q}$ and its degree over $\mathbb{Q}$, justifying all steps.
   (b) Determine whether or not $\text{Gal}(L/\mathbb{Q})$ is abelian.
   (c) Determine whether or not $\text{Gal}(L/\mathbb{Q})$ is a solvable group, and if so, give an appropriate normal tower which demonstrates this fact. If not, be clear why the extension fails to have a solvable Galois group.

Topology

1. Let $X$ and $Y$ be topological spaces with $x_0 \in X$ and $y_0 \in Y$. Let $X \times Y$ have the product topology. Show that $\pi(X \times Y, (x_0, y_0))$ is isomorphic to $\pi(X, x_0) \times \pi(Y, y_0)$.

2. Let $M$ be a smooth manifold, $X$ a continuous vector field on $M$ (i.e., a continuous section of the tangent bundle $TM$). There are two reasonable definitions of what it means for $X$ to be smooth at a point $p$ in $M$:

   (a) Definition 1: Let $(x, U)$ be a local coordinate system defined on an open neighborhood $U$ of $p$; then $X$ can be expressed in local coordinates as $X = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}$ for some real-valued functions $a^1, \ldots, a^n$ defined on $U$. Then $X$ is smooth at $p$ provided that each coefficient function $a^i$ is smooth at $p$.
   (b) Definition 2: The vector field $X$ is smooth at $p$ if for every smooth function $f$ defined on a neighborhood of $p$, the function $X(f)$ is smooth at $p$.

   Prove that these two definitions are equivalent.

3. Show that $S^{n-1}$ is not a retract of $E^n = \{ x \in \mathbb{R}^n : |x| \leq 1 \}$ for $n \geq 1$. Use this to prove the Brouwer Fixed-Point Theorem; that is, show that if $n \geq 1$, then any continuous map $f : E^n \to E^n$ must have a fixed point.

4. a) Does a boundary of a parallelizable manifold have to be a parallelizable manifold? Prove your answer.
   b) Does a product of two parallelizable manifolds have to be a parallelizable manifold? Prove your answer.
   c) Is the Klein bottle a parallelizable manifold? How about the torus $S^1 \times S^1$? Prove your answer.
5. Let \( n \geq 2 \) and \( B \subset S^n \) be a wedge of two circles; that is, \( B \) is a closed subset of \( S^n \) homeomorphic to a figure eight so that \( B = C \cup D \) with \( C \) and \( D \) homeomorphic to \( S^1 \) and \( C \cap D \) a single point. Compute \( H_q(S^n \setminus B) \) for \( n \geq 2 \).

6. a) Let \( \phi : S^2 \to \mathbb{R}^{17} \) be a smooth map. Let \( \omega \) be a closed 2-form on \( \mathbb{R}^{17} \). Compute the integral \( \int_{S^2} \phi^* \omega \).

   b) Let \( \phi : S^3 \to S^2 \) and \( \psi : S^2 \to S^4 \) be smooth maps of oriented manifolds. Let \( \omega \) be a 3-form on \( S^4 \). Compute \( \int_{S^3} (\psi \circ \phi)^* \omega \).

Analysis

1. Suppose \( f \) is entire and \( \lim_{z \to \infty} f(z) \in \mathbb{C} \) exists. Show that \( f \) is constant.

2. Let \((V, (\cdot, \cdot))\) be an inner product space over the field \( \mathbb{F} \).

   a.) If \( \mathbb{F} = \mathbb{R} \), show that vectors \( x, y \in V \) are orthogonal if and only if

   \[
   \|x + y\|^2 = \|x\|^2 + \|y\|^2.
   \]

   b.) Show that (a) is false for any complex \((\mathbb{F} = \mathbb{C})\) inner product space \( V \), where \( x \) can be any nonzero vector in \( V \). (Hint: \( y \) should be more imaginary than \( x \).)

3. In each of the following, you are given a domain \( D \) and a function \( f : D \to \mathbb{C} \). Determine whether \( f \) has an anti-derivative on \( D \).

   (a) \( f(z) = e^{1/z} \log(z) \) where \( D \) is the complex plane with the origin and negative real axis removed.

   (b) \( f(z) = \frac{1}{z^2 - 1} \) where \( D \) consists of all points in \( \mathbb{C} \) except for \( \pm 1 \).

   (c) \( f(z) = \exp\left(\frac{1}{z^2}\right) \), where \( D = \mathbb{C} \setminus \{0\} \).

4. Consider \( C[0, 1] \) with the uniform norm \( \|f\|_\infty = \sup_{x \in [0, 1]} |f(x)| \). Show that the linear map

   \[
   V : C[0, 1] \to C[0, 1]
   \]

   defined by the formula

   \[
   V(f)(x) = \int_0^x f(t) \, dt
   \]

   is a bounded linear operator with no eigenvalues.

5. Find the limit of each of the following sequences of integrals. Justify fully. (Here \( m \) denotes Lebesgue measure on \( \mathbb{R} \).)
6. Let $f, g$ be $2\pi$-periodic (Lebesgue) measurable functions on $\mathbb{R}$. Let $f \ast g$ denote the (normalized) convolution function

$$f \ast g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) g(x - t) \, dt.$$ 

a.) Show that if (their restrictions) $f, g \in L^2[-\pi, \pi]$ then $f \ast g(x)$ exists and is bounded on $[-\pi, \pi]$, in fact,

$$\|f \ast g\|_{\infty} = \sup_{x \in [-\pi, \pi]} |f \ast g(x)| \leq \frac{1}{2\pi} \|f\|_2 \|g\|_2.$$

b.) Show also that $\hat{f} \ast g(n) = \hat{f}(n) \hat{g}(n)$ for all $n \in \mathbb{Z}$, where

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$$

is the $n$-th Fourier coefficient of $f$ for $n \in \mathbb{Z}$. 

(a) \( \lim_{n \to \infty} \int_{[0,\infty)} f_n \, dm \) where \( f_n(x) = \frac{\sin(nx)}{n(1 + x^2)} \)

(b) \( \lim_{n \to \infty} \int_{[0,\infty)} f_n \, dm \) where \( f_n(x) = e^{-\frac{x}{n}} \frac{1}{1 + x} \).