1. Let \( A \) be a \( 5 \times 5 \) matrix over \( \mathbb{C} \) with minimal polynomial \( m_A(x) = x^2(x - 2)^2 \). What are the possible rational canonical forms and corresponding Jordan forms for \( A \)?

2. Let \( R \) be a commutative ring with identity.
   (a) Let \( M, N \) be free \( R \)-modules. Show that \( M \otimes_R N \) is free.
   (b) Let \( M, N \) be projective \( R \)-modules. Show that \( M \otimes_R N \) is projective.

3. Suppose that \( p \) and \( q \) are distinct primes and that \( G \) is a group of order \( p^2q \). Show that \( G \) has either a normal \( p \)-Sylow subgroup or a normal \( q \)-Sylow subgroup.

4. Let \( \zeta_7 \in \mathbb{C} \) denote a primitive, complex 7th root of unity. Consider the lattice of fields:

\[
\begin{align*}
L &= \mathbb{Q}(\sqrt[7]{2}, \zeta_7) \\
E &= \mathbb{Q}(\zeta_7) \\
K &= \mathbb{Q}(\sqrt[7]{2}) \\
\mathbb{Q} &\quad \text{(root at the bottom)}
\end{align*}
\]

(a) Compute the degree of each extension in the diagram, justifying your answers.

(b) Show that \( L \) is Galois over \( \mathbb{Q} \). Let \( G = \text{Gal}(L/\mathbb{Q}) \) and \( H_E, H_K \) be the subgroups corresponding to \( E \) and \( K \), respectively. Show explicitly that \( H_E \) and \( H_K \) are cyclic groups and compute their orders. Using the Galois correspondence, determine the subgroups \( H_E \cap H_K \) and \( H_E H_K \).

(c) Show that \( G \) is a semidirect product of cyclic groups.

(d) Let \( \sigma \in G \) be characterized by \( \sigma(\sqrt[7]{2}) = \sqrt[7]{2}\zeta_7^5 \) and \( \sigma(\zeta_7) = \zeta_7^3 \). What are the fixed fields corresponding to \( \sigma H_E \sigma^{-1} \) and \( \sigma H_K \sigma^{-1} \)?

5. (a) Let \( K/\mathbb{Q} \) be a field extension of degree 24. Show that \( x^5 + 2x^4 - 16x^3 + 6x - 10 \) has no roots in \( K \).

(b) Show that \( \alpha = \sqrt[5]{2} + \sqrt[5]{5} \) is algebraic over \( \mathbb{Q} \), and determine the degree of \( \alpha \).

6. (a) Determine the number of distinct roots of the polynomial \( x^n - 1 \) in an algebraic closure of \( \mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \), where \( p \) is a prime number and \( n > 0 \).

(b) Let \( K/\mathbb{Q} \) be a finite extension of fields, and let \( \alpha \in K \). Suppose that there is a monic polynomial \( f \in \mathbb{Z}[x] \) so that \( f(\alpha) = 0 \). Show that the minimal polynomial \( m_{\alpha, \mathbb{Q}}(x) \) of \( \alpha \) over \( \mathbb{Q} \) lies in \( \mathbb{Z}[x] \).
1. Let \((X, \mathcal{M}, \mu)\) be a measure space. Let \(\{E_n\}_{n=1}^{\infty}\) be a sequence on \(\mathcal{M}\) such that \(E_1 \subset E_2 \subset E_3 \ldots\) and let

\[ E = \bigcup_{n=1}^{\infty} E_n. \]

Also let \(\{A_n\}_{n=1}^{\infty}\) be a sequence on \(\mathcal{M}\) such that \(A_1 \supset A_2 \supset A_3 \ldots\) and let

\[ A = \bigcap_{n=1}^{\infty} A_n. \]

(a) Suppose that \(f : X \to \mathbb{R}\) is integrable. Show that

\[ \int_E f \, d\mu = \lim_{n \to \infty} \int_{E_n} f \, d\mu. \]  

(1)

and

\[ \int_A f \, d\mu = \lim_{n \to \infty} \int_{A_n} f \, d\mu. \]  

(2)

(b) Suppose that \(f \in L^+(X, \mathcal{M}, \mu)\), i.e., \(f\) is a non-negative measurable real-valued function but is not necessarily integrable. Show that Equation (1) is valid for \(f\).

(c) Give an example of a measure space \((X, \mathcal{M}, \mu)\) and a function \(f \in L^+(X, \mathcal{M}, \mu)\) such that Equation (2) fails to hold for \(f\).

2. Suppose that \((X, \mathcal{M}, \mu)\) is a measure space satisfying \(\mu(X) < \infty\). Show that \(L^q(X, \mathcal{M}, \mu) \subset L^p(X, \mathcal{M}, \mu)\) whenever \(0 < p < q\). (Be sure to include the case that \(q = \infty\).)

3. Let \(\Gamma^+_R\) be the semicircle defined by \(|z| = R\) and \(\text{Im}(z) \geq 0\), and let \(\Gamma^-_R\) be the semicircle given by \(|z| = R\) and \(\text{Im}(z) \leq 0\). Give both semicircles the counterclockwise orientation.

(a) Evaluate

\[ \lim_{R \to \infty} \int_{\Gamma^+_R} \frac{e^{iz}}{z^4} \, dz. \]

(b) Evaluate

\[ \lim_{R \to \infty} \int_{\Gamma^-_R} \frac{e^{iz}}{z^4} \, dz. \]

4. Let \((V, \| \cdot \|_V)\) and \((W, \| \cdot \|_W)\) be normed vector spaces. Give the Cartesian product

\[ V \times W = \{(x, y) \mid x \in V, y \in W\} \]
the obvious coordinate-wise defined vector space structure and the norm
\[ \|(x, y)\|_{V \times W} = \|x\|_V + \|y\|_W \quad \text{for } x \in V, y \in W. \]

Prove that the graph
\[ G(T) = \{(x, T(x)) : x \in V\} \]
of a continuous, linear mapping \( T : V \to W \) is a closed, linear subspace of \((V \times W, \|\cdot\|_{V \times W}).\)

5. Let \( c \) denote the \( \mathbb{C} \)-vector space of all convergent complex sequences. Show that \( c \) is a Banach space when equipped with the supremum norm from \( \ell^\infty:\)
\[ \|(x_n)\|_\infty = \sup_{n \in \mathbb{N}} |x_n|. \]

6. Let \( \mathcal{H} \) be a separable infinite-dimensional Hilbert space. Consider the set
\[ F(\mathcal{H}) = \{T \in B(\mathcal{H}) : \dim(\text{range}(T)) < \infty\} \]
of bounded finite rank operators. It is easy to see that \( F(\mathcal{H}) \) is a subalgebra of the algebra \( B(\mathcal{H}) \), but it has further structure.

a.) Show that if \( T \in F(\mathcal{H}) \) then \( T^* \in F(\mathcal{H}) \) and \( \dim(\text{range}(T^*)) = \dim(\text{range}(T)). \)

b.) Show that if \( T \in F(\mathcal{H}) \) then \( ST, TS \in F(\mathcal{H}) \) for all \( S \in B(\mathcal{H}). \)
Topology

1. Let \( C : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3 \) by \( C(v, w) = v \times w \), the usual vector cross product. Determine the critical points of \( C \). Conclude that for \( 0 \neq u \in \mathbb{R}^3 \), the set \( \{(v, w) \in \mathbb{R}^3 \times \mathbb{R}^3 : v \times w = u\} \) is a smooth manifold. If \( \{e_1, e_2, e_3\} \) denotes the standard basis for \( \mathbb{R}^3 \), determine a basis of the tangent space \( T_{(e_1,e_2)}(C^{-1}(e_3)) \) as a vector subspace of \( T_{(e_1,e_2)}(\mathbb{R}^3 \times \mathbb{R}^3) \cong \mathbb{R}^6 \).

2. Let \( p : Y \to X \) be a covering map. Let \( Z \) be any connected space, and let \( f : Z \to X \) be a continuous map. Suppose that \( f_1 : Z \to Y \) and \( f_2 : Z \to Y \) are continuous lifts of \( f \) (i.e., \( p \circ f_i = f \) for \( i = 1, 2 \)) that agree at some point \( z_0 \in Z \). Show that \( f_1 = f_2 \) on all of \( Z \).

3. Consider the circle \( S^1 \) with its usual \( CW \)-structure with a single 0-cell \( e^0 \) and a single 1-cell \( e^1 \). Let \( X \) be the space obtained from \( S^1 \) by attaching two 2-cells \( e^2_1 \) and \( e^2_2 \) by maps of degree 2 and degree 3, respectively. Compute the homology groups of \( X \).

4. Let \( f \in \mathbb{R}[X,Y,Z] \) be a homogeneous quadratic polynomial with real coefficients. Let \( D^3 = \{x \in \mathbb{R}^3 : \|x\| \leq 1\} \) be the unit disk and \( S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\} \) the unit sphere in Euclidean space \( \mathbb{R}^3 \). Let \( \nu \) denote the volume form on \( S^2 \), where \( S^2 \) is given the orientation induced from the standard orientation of \( D^3 \). Prove that

\[
\int_{D^3} \Delta f = 2 \int_{S^2} f \nu,
\]

where \( \Delta f \) is the Laplacian \( \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \). [Hint: Write \( \Delta f \) as a divergence.]

5. Let \( \mathbb{R}P^n \) denote real projective \( n \)-space, the quotient space of \( \mathbb{R}^{n+1} - \{0\} \) obtained from the equivalence relation identifying two nonzero vectors if they span the same line. Show that if \( n > 0 \) is even, then every continuous map \( f : \mathbb{R}P^n \to \mathbb{R}P^n \) has a fixed point. [Hint: Translate the problem to one about mappings \( S^n \to S^n \); recall the proof that an even-dimensional sphere has no nowhere-vanishing vector field. You may use without proof that \( S^n \) is simply connected for \( n > 1 \).]

6. Let \( M \) be a smooth \( n \)-manifold whose smooth structure is defined by a maximal atlas \( \mathcal{M} \) of charts \( (x, U) \), where \( U \subseteq M \) is open and \( x : U \to x(U) \subseteq \mathbb{R}^n \) is a homeomorphism of \( U \) with an open subset of \( \mathbb{R}^n \). Suppose that there is a nowhere-vanishing smooth \( n \)-form \( \omega \in \Omega^n(M) \). Show that there is a subcollection \( \mathcal{A} \) of \( \mathcal{M} \) such that

- The collection \( \{U : (x, U) \in \mathcal{A}\} \) is an open cover of \( M \).
- For any two overlapping charts charts \( (x, U), (y, V) \in \mathcal{A} \), i.e., any two charts in \( \mathcal{A} \) such that \( U \cap V \neq \emptyset \), the derivative \( D(y \circ x^{-1})(x(p)) : \mathbb{R}^n \to \mathbb{R}^n \) of the mapping \( y \circ x^{-1} : x(U \cap V) \to y(U \cap V) \) has positive determinant for every \( p \in U \cap V \).