September 2016 Written Certification Exam

Algebra

1. A linear operator $E$ on a finite-dimensional $k$-vector space $V$ is a projection if $V$ admits a direct sum decomposition $V = U \oplus W$ where $U$ and $W$ are $E$-invariant subspaces such that $E|_U = \text{Id}$ and $E|_W = 0$.

(a) Let $E, F \in \text{End}_k(V)$ be projections. Show that if $E$ and $F$ commute, then $EF$ is a projection.

(b) Is the converse true? That is, if $E$ and $F$ are projections such that $EF$ is also a projection, must $E$ and $F$ commute?

2. Consider the group $\text{SL}_2(F_3)$ of $2 \times 2$ matrices of determinant 1 over the 3-element field $F_3$.

(a) Determine the order of $\text{SL}_2(F_3)$.

(b) How many 3-Sylow subgroups does $\text{SL}_2(F_3)$ have? What is an example? What is the structure of its normalizer?

3. Let $k$ be a field, let $V$ be a finite-dimensional $k$-vector space, and let $T : V \rightarrow V$ be a linear operator on $V$. Show that $V$ admits a direct sum decomposition $V = U \oplus W$, where $U$ and $W$ are subspaces satisfying

(a) $T(U) \subseteq U$ and $T(W) \subseteq W$.

(b) The restriction $T|_U : U \rightarrow U$ is nilpotent.

(c) The restriction $T|_W : W \rightarrow W$ is invertible.

4. Unique Factorization.

(a) Show that $\mathbb{Z} [\sqrt{-5}]$ is a Noetherian integral domain, but not a UFD.

(b) Show that a Noetherian integral domain in which every irreducible element is a prime element is a UFD.

5. Let $L$ be the splitting field of $x^9 - 5^3$ over $\mathbb{Q}$, and $\zeta_9$ a primitive 9th root of unity in $\mathbb{C}$.

(a) Determine the degree $[L : \mathbb{Q}]$, giving reasons to support your statements.

(b) Determine the isomorphism classes of $\text{Gal}(L/\mathbb{Q}(\zeta_9))$ and $\text{Gal}(L/\mathbb{Q}(\sqrt[3]{5}))$.

(c) Show that $G = \text{Gal}(L/\mathbb{Q})$ has a normal subgroup $H$ with $G/H \cong S_3$, the symmetric group on 3 letters.

6. Let $f \in \mathbb{Q}[x]$ be a polynomial of degree $n \geq 3$, and let $K$ be the splitting field of $f$ over $\mathbb{Q}$. Suppose that $\text{Gal}(K/\mathbb{Q}) \cong S_n$, the symmetric group.
(a) Show that \( f \) is irreducible.

(b) If \( \alpha \) is a root of \( f \) in \( K \), show that \( \text{Aut}(\mathbb{Q}(\alpha)/\mathbb{Q}) \) is trivial, that is show that every automorphism of \( \mathbb{Q}(\alpha) \) which fixes \( \mathbb{Q} \) pointwise is the identity.

(c) If \( n \geq 4 \), show that \( \alpha^n \not\in \mathbb{Q} \).
1. Let \((X, \mathcal{M}, \mu)\) be a measure space and let \(g \in L^+(X, \mathcal{M})\), i.e., \(g : X \to [0, \infty]\) is a measurable function. Define \(\nu : \mathcal{M} \to [0, \infty]\) by
\[
\nu(E) = \int_E g \, d\mu.
\]
(a) Show that \(\nu\) is a measure on \((X, \mathcal{M})\).
(b) Show that for any nonnegative measurable function \(f : X \to [0, \infty]\), we have
\[
\int_X f \, d\nu = \int_X fg \, d\mu.
\]
(c) Suppose that \(g \in L^2(X, \mathcal{M}, \mu)\). Show that
\[
L^2(X, \mathcal{M}, \mu) \subset L^1(X, \mathcal{M}, \nu).
\]

2. Let \(m^*\) be Lebesgue outer measure on \(\mathbb{R}\). Let \(E\) be a subset of \(\mathbb{R}\) with \(m^*(E) < \infty\).
   (a) Show that there exists a Borel set \(B\) such that \(E \subset B\) and \(m^*(E) = m^*(B)\).
   (b) Let \(B\) be as in part (a). Show that \(E\) is Lebesgue measurable if and only if \(m^*(B - E) = 0\). (You may use the facts that Borel sets are Lebesgue measurable and that Lebesgue measure is complete.)

3. (a) State the Cauchy Integral Formulas for analytic functions \(f\) and their derivatives \(f^{(n)}(z)\).
   (b) Use the Cauchy Integral Formulas to derive the Cauchy estimates which give bounds for \(f\) and its derivatives at a point \(z_0\) in terms of the maximum of \(f\) on a circle \(|z - z_0| = R\). Be sure to state any hypotheses.
   (c) State and prove Liouville’s Theorem.

4. Suppose that \(T : X \to Y\) is a linear map between Banach spaces. Show that \(T\) is bounded if and only if \(T\) is continuous at 0.
5. Let $T : H \to H$ and $S : H \to H$ be functions (not necessarily linear) from a Hilbert space $H$ to itself. Suppose that for each $x, y \in H$ we have

$$(T(x) \mid y) = (x \mid S(y)).$$

Show that $T$ and $S$ are bounded linear maps with $S = T^*$.

6. We say that $f : [0, 1] \to \mathbb{R}$ is $\alpha$-Hölder if

$$h_\alpha(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^{\alpha}} < \infty.$$ 

For $M > 0$ and $\alpha \in (0, 1]$, let

$$H_{\alpha,M} := \{ f \in C([0, 1]) : h_\alpha(f) \leq M \text{ and } \|f\|_\infty \leq M \}.$$ 

Show that $H_{\alpha,M}$ is compact in $C([0, 1])$. 

1. Show that the group SL(n, \mathbb{R}) consisting of all $n \times n$ matrices with determinant 1 is a smooth manifold.

2. Let $p \in M$ be a point in a smooth manifold $M$ and let $\mathcal{F}_p$ be the subspace of $C^\infty(M)$ consisting of all smooth functions that vanish at $p$. Let $\mathcal{F}_p^2 \subset \mathcal{F}_p$ be the subspace spanned by functions of the form $fg$ for $f, g \in \mathcal{F}_p$. Define a map $\Phi : \mathcal{F}_p \to T_p^*M$ by setting

$$\Phi(f) = df_p.$$ 

Show that the restriction of $\Phi$ to $\mathcal{F}_p^2$ is zero, and that $\Phi$ descends to an isomorphism $\mathcal{F}_p/\mathcal{F}_p^2 \to T^*_pM$ of vector spaces.

**Hint:** You can use the following fact without a proof. Let $\phi$ be a local chart centered at $p$ (i.e., $\phi$ is a chart defined on a neighborhood of $p$ such that $\phi(p) = 0$); then $f \in \mathcal{F}_p$ if and only if $(f \circ \phi^{-1})(x_1, \ldots, x_n) = \sum_{i=1}^n x_i f_i(x_1, \ldots, x_n)$ for some smooth functions $f_i$.

3. Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ be the torus defined by

$$T^2 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 = 1, z^2 + t^2 = 1\}$$

with the orientation determined by its product structure, where each circle factor $S^1 \subseteq \mathbb{R}^2$ is oriented as the boundary of the unit disk. Compute $\int_{T^2} zd\mathbf{x} \wedge dt$.

4. Let $A$ be the curve inside of a solid torus $S^1 \times D^2$ pictured in the figure below. Show that there is no retraction of the solid torus onto $A$. 

![Diagram of a curve A inside a solid torus](image-url)
5. Let $X$ be the quotient space of $S^2$ under the identification $x \sim -x$ for $x$ in the equator of $S^2$. Compute the homology groups $H_i(X; \mathbb{Z})$ for all $i$.

6. Show that for finite $CW$-complexes $X$ and $Y$, the Euler characteristic $\chi$ satisfies

$$\chi(X \times Y) = \chi(X) \times \chi(Y).$$