1. Let $V$ be a finite-dimensional vector space over a field $k$, and let $T : V \to V$ be a linear operator. Show that there is a decomposition $V = V_i \oplus V_n$ of $V$ into $T$-invariant subspaces such that $T|_{V_i} : V_i \to V_i$ is invertible and $T|_{V_n} : V_n \to V_n$ is nilpotent.

2. Let $A$ be a commutative ring, and let $S$ and $T$ be multiplicatively closed subsets of $A$ such that $1 \in S \subseteq T$.

(a) Show that there is a well defined ring homomorphism $f : S^{-1}A \to T^{-1}A$ defined by $\frac{a}{s} \mapsto \frac{a}{s}$.

(b) Show that the following are equivalent:
   i. The map $f$ is an isomorphism.
   ii. For each $t \in T$, the element $\frac{t}{1} \in S^{-1}A$ is a unit in $S^{-1}A$.
   iii. For each $t \in T$, there exists $x \in A$ such that $xt \in S$.

3. Show that the group $\text{SL}_2(\mathbb{F}_4)$ of two by two matrices of determinant one over the four-element field $\mathbb{F}_4$ is isomorphic to the alternating group $A_5$.

4. In this problem, each part is independent of the others. For each part, give a short but complete answer.

(a) Is $f(x) = x^3 + x + 1$ irreducible over $\mathbb{F}_{256}$?

(b) Let $\alpha = \sqrt[3]{10 + 6\sqrt{3}} \in \mathbb{R}$, i.e., $\alpha \in \mathbb{R}_{>0}$ satisfies $\alpha^3 = 10 + 6\sqrt{3} > 0$. Is the extension $\mathbb{Q}(\alpha)$ Galois over $\mathbb{Q}$?

(c) Let $K \supseteq F$ be an algebraic field extension, not necessarily finite. Let $R \subseteq K$ be a ring such that $F \subseteq R$. Show that $R$ is a field.

5. Recall that a quartic polynomial $f(x) = x^4 + ax^2 + bx + c$ has discriminant
   \[ D(f) = 16a^4c - 4a^3b^2 - 128a^2c^2 + 144ab^2c - 27b^4 + 256c^3 \]
and resolvent cubic
   \[ g(x) = x^3 - ax^2 - 4cx + (4ac - b^2). \]

(a) For all primes $p \neq 3, 5$, determine the Galois group of $f_p(x) = x^4 + px + p$ over $\mathbb{Q}$.

(b) Determine the Galois group of $f_3(x) = x^4 + 3x + 3$. [Hint: reduce modulo $13$.]

6. Let $\zeta = e^{2\pi i/15} \in \mathbb{C}$.

(a) Let $f(x) \in \mathbb{Q}[x]$ be the minimal polynomial of $\zeta$ over $\mathbb{Q}$. What is the degree of $f(x)$? List the zeros of $f(x)$.

(b) Show that $K = \mathbb{Q}(\zeta)$ is Galois over $\mathbb{Q}$.

(c) Let $G := \text{Gal}(K | \mathbb{Q})$. Exhibit an element $\sigma \in G$ of order 4.

(d) Consider $\alpha = \zeta + \zeta^{-1} = 2\cos(2\pi/15)$. Is the field $\mathbb{Q}(\alpha)$ constructible?
1. Use the dominated convergence theorem (DCT) to prove the following:

**Theorem** Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be differentiable and \( f' \) be bounded. Then \( f' \) is Lebesgue integrable on a closed interval \([a, b]\) and

\[
\int_a^b f'(x) d\lambda = f(b) - f(a).
\]

**Hint:** Consider the functions \( g_n(x) := n \cdot (f(x + \frac{1}{n}) - f(x)) = \frac{f(x + \frac{1}{n}) - f(x)}{\frac{1}{n}} \) for \( n \in \mathbb{N} \).

2. Let \( \gamma \) be a simple closed path around a bounded convex domain \( \Omega \subset \mathbb{C} \). Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be holomorphic. Show that

\[
\int_{\gamma} \frac{f'(w)}{w - z} \, dw = \int_{\gamma} \frac{f(w)}{(w - z)^2} \, dw \quad \text{for all} \quad z \in \mathbb{C} \setminus \{\gamma\}
\]

3. (Liouville’s theorem)
   a) State Liouville’s theorem.
   b) Let \( f, g : \mathbb{C} \rightarrow \mathbb{C} \) be entire functions, such that

\[
|f(z)| \leq |g(z)| \quad \text{for all} \quad z \in \mathbb{C}.
\]

Show that \( f(z) = c \cdot g(z) \) for some constant \( c \in \mathbb{C} \).

**Hint:** Consider the function \( \frac{f}{g} \).

4. Let \( X \) be a complex Banach space and \( \phi : X \rightarrow \mathbb{C} \) a linear map. Show that \( \phi \) is bounded if and only if \( \phi \) is continuous at \( 0 \).

5. Let \( A \) be the collection \( C([0, 1]) \) of continuous complex-valued functions on \([0, 1]\). Show that \( A \) is complete with respect to the metric \( \rho(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)| \).

6. Let \( H \) be a complex Hilbert space and \( T : H \rightarrow H \) a linear map. Show that \( T \) is bounded if and only if \( T \) is continuous from \( H \) to \( H \) when \( H \) is given the weak topology.
1. Let $M$ and $N$ be smooth manifolds.

(a) Let $A \subseteq M$ be an arbitrary non-empty subset. What does it mean for a function $F : A \rightarrow N$ to be smooth?

(b) Let $A \subseteq M$ be a non-empty closed subset of $M$, $F : A \rightarrow \mathbb{R}^k$ a smooth function, and $O \subseteq M$ an open subset of $M$ containing $A$. Please show there exists a smooth function $\tilde{F} : M \rightarrow \mathbb{R}^k$ such that the restriction of $\tilde{F}$ to $A$ agrees with $F$ and the support of $\tilde{F}$ is contained in $O$.

2. Let $M$ be a smooth manifold and recall that given a smooth diffeomorphism $F : M \rightarrow M$ and a smooth vector field $X$ on $M$, we obtain a smooth vector field $Y = F_*(X)$ on $M$ via $Y_p \equiv F_*(X_{F^{-1}(p)})$. Now, let $G$ be a Lie group and $\nu : G \rightarrow G$ be given by $x \mapsto x^{-1}$. Please demonstrate that for any left-invariant vector field $X$ on $G$, the vector field $Y \equiv \nu_*(X)$ is right-invariant.

3. Let $\omega$ be the smooth 1-form on $\mathbb{R}^3$ given by

$$\omega = (e^{y+z} - 2y)dx + (xe^{y+z} + y)dy + e^{x+y}dz,$$

and let $S \subset \mathbb{R}^3$ be graph of the function $f(x, y) = x^2 + y^2$ over the unit disk:

$D = \{(x, y) : x^2 + y^2 \leq 1\}$. Please compute

$$\int_S d(i^*\omega).$$

4. Let $X = \mathbb{R}^3 \setminus (C \cup L)$ be the manifold obtained from $\mathbb{R}^3$ by removing the circle $C = \{(\cos t, \sin t, 0) \mid 0 \leq t < 2\pi\}$ and the $z$-axis $L = \{(0, 0, z) \mid z \in \mathbb{R}\}$.

What is the fundamental group of $X$?

5. (a) Choose a $\Delta$-complex structure for the real projective plane $\mathbb{R}P^2$.

(b) Compute the cohomology ring $H^\bullet(\mathbb{R}P^2; \mathbb{Z}_2)$, and show that it is isomorphic to $\mathbb{Z}_2[x]/(x^3)$, where $x$ is an element of degree 1.

6. Prove that every continuous map $f : S^2 \rightarrow S^1 \times S^1$ has degree zero.