Properties of Groupoid Dynamical Systems and Associated Crossed Product C^*-algebras

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Overview

The groupoid crossed product directly generalizes the idea of a crossed product C^*-algebra, which arises when a locally compact group acts on a C^*-algebra. A great deal is known about these older objects, and one can ask whether certain results generalize to the groupoid case. One major result, due to Philip Green, says that if A is nuclear and G is amenable, the crossed product $A \rtimes G$ is nuclear. We present the analogue for groupoids here. We also discuss a weaker property of C^*-algebras, called exactness.

Groupoid Dynamical Systems

Basics

Informally, a **groupoid** is a set G with a partially defined multiplication operation and an inversion map $G \to G$:

$$(x, y) \mapsto xy, \quad x \mapsto x^{-1}.$$

The **unit space** consists of all the elements that behave like identities:

$$G^{(0)} = \{u \in G : u = u^{-1} = u^2\}.$$

There are **range** and **source** maps $r, s : G \to G^{(0)}$:

$$r(x) = xx^{-1}, s(x) = x^{-1}x,$$

We often view the groupoid as fibred over $G^{(0)}$ by r or s:

$$G^u = r^{-1}(u), \quad G_u = s^{-1}(u).$$

One can think of a groupoid as a group in which the product is only partially defined. Indeed, any group is an example of a groupoid. A less trivial example would be a bundle of groups.

Groupoid Actions

A groupoid G can act on a set X, provided that X is fibred over $G^{(0)}$ - i.e., if there is a surjection $r_X : X \to G^{(0)}$. The action is much like a group action, except it is only partially defined: if $x \in G$ and $z \in X$, then $x \cdot z$ is defined exactly when $s(x) = r_X(z)$.

Dynamical Systems

Let G be a locally compact Hausdorff groupoid. For G to act on a C^*-algebra A, we need A to be fibred over $G^{(0)}$. We require that A be a $C_0(G^{(0)})$-algebra; that is, there is a bundle $\mathcal{A}$ of C^*-algebras over $G^{(0)}$, and A can be viewed as a section algebra of $\mathcal{A}$. An **action** of G on A is just a family of isomorphisms

$$\alpha = \{\alpha_x\}_{x \in G}, \quad \alpha_x : \mathcal{A}_{s(x)} \to \mathcal{A}_{r(x)}.$$

The triple (A, G, α) is called a **groupoid dynamical system**.

Crossed Products

The Full Crossed Product

Given a groupoid dynamical system (A, G, α), one can build a new C^*-algebra that encodes information about the system. We start by forming the pullback bundle

$$r^*\mathcal{A} \to G$$

and we consider the space of continuous compactly supported sections

$$\Gamma_c(G, r^*\mathcal{A}).$$

We assume that G carries a **Haar system** - a family of measures which plays the role of the Haar measure on a locally compact group. Using these measures, it's possible to define a convolution-like product and an involution which make $\Gamma_c(G, r^*\mathcal{A})$ into a $*$-algebra. We can define a norm on it by considering an appropriate collection of $*$-representations of $\Gamma_c(G, r^*\mathcal{A})$ on Hilbert space. The completion is a C^*-algebra,

$$A \rtimes_\alpha G,$$

called the **crossed product** of A by G.

The groupoid crossed product directly generalizes the idea of a crossed product by a *group* (which in turn generalizes group C^*-algebras). Indeed, if G is locally compact group, then the groupoid crossed product associated to G agrees with the usual one.

The Reduced Crossed Product

The full crossed product is relatively nice in theory, but it can be mysterious in practice. We can build a related object,

$$A \rtimes_{\alpha,r} G,$$

called the **reduced crossed product**, which is slightly nicer. It is a quotient of $A \rtimes_\alpha G$: any faithful representation

$$\pi : A \to B(\mathcal{H})$$

on a Hilbert space $\mathcal{H}$ induces a representation $\operatorname{Ind}\pi$ of $A \rtimes_\alpha G$, which may not be faithful. We then have

$$A \rtimes_{\alpha,r} G = A \rtimes_\alpha G / \ker(\operatorname{Ind}\pi).$$

This construction is independent of the choice of π.

Amenability

It would be nice if the full and reduced crossed products agreed. This isn't always the case, but there are conditions under which it is true.

Proposition 1 (Sims-Williams, 2012 [4]) If G is an amenable groupoid, then $A \rtimes_\alpha G = A \rtimes_{\alpha,r} G$.

There are several notions of amenability for groupoids; the form used here is measure-theoretic. Each is technical, and they all reduce to the usual definition of amenability when G is a group.

Nuclearity and the Main Theorem

Nuclear C^*-algebras

Nuclearity is a very desirable condition for a C^*-algebra to have. In short, nuclear C^*-algebras behave well with respect to tensor products. If A and B are C^*-algebras, the *algebraic* tensor product

$$A \odot B$$

may carry more than one C^*-norm. Therefore, there are generally multiple ways to complete it into a tensor product C^*-algebra. Most people care about the two extremes:

$$A \otimes_{\max} B \quad \text{and} \quad A \otimes_\sigma B,$$

the maximal and minimal (or spatial) tensor products. If the various tensor products coincide, then the situation is very nice.

Definition A C^*-algebra A is called **nuclear** if

$$A \otimes_{\max} B = A \otimes_\sigma B$$

for every C^*-algebra B.

Most reasonably nice C^*-algebras (such as commutative ones) are nuclear.

Nuclearity for Crossed Products

It was shown by Philip Green in [2] that if A is a nuclear C^*-algebra and G is an amenable *group*, then $A \rtimes_\alpha G$ is nuclear. We present the analogue for groupoids here.

Theorem 1 (L., 2012) If (A, G, α) is a groupoid dynamical system with A nuclear and G amenable, then the crossed product $A \rtimes_\alpha G$ is nuclear.

The idea of the proof is similar to Green's original one for group crossed products.

- For any C^*-algebra B, there is a canonical surjection
  $$\kappa : (A \rtimes_\alpha G) \otimes_{\max} B \to (A \rtimes_\alpha G) \otimes_\sigma B.$$
  It suffices to show that κ is injective.
- The first step is to verify that
  $$(A \rtimes_\alpha G) \otimes_{\max} B \cong (A \otimes_{\max} B) \rtimes_{\alpha\otimes\mathrm{id}} G$$
  in a natural way. A similar result holds for reduced crossed products:
  $$(A \rtimes_{\alpha,r} G) \otimes_\sigma B \cong (A \otimes_\sigma B) \rtimes_{\alpha\otimes\mathrm{id},r} G.$$

Outline of Proof

The rest of the proof can now be summed up by the following commutative diagram:

$$\begin{array}{ccc} (A \rtimes_\alpha G) \otimes_{\max} B & \xrightarrow{\sim} & (A \otimes_{\max} B) \rtimes_{\alpha\otimes\mathrm{id}} G \\ \downarrow & & \downarrow \\ (A \rtimes_{\alpha,r} G) \otimes_\sigma B & \xrightarrow{\sim} & (A \otimes_\sigma B) \rtimes_{\alpha\otimes\mathrm{id},r} G \end{array}$$

Since A is nuclear and G is amenable, the right side is an isomorphism (in fact, the identity map). Therefore, the left side is also an isomorphism. But

$$(A \rtimes_{\alpha,r} G) \otimes_\sigma B = (A \rtimes_\alpha G) \otimes_\sigma B,$$

and it can be checked that this map is just κ. It follows that $A \rtimes_\alpha G$ is nuclear.

We've glossed over many technical hurdles that result from working with groupoids, but the proof comes down to verifying certain facts about this diagram.

Exactness and Further Directions

A concept which is related to (though weaker than) nuclearity is **exactness**.

Definition A C^*-algebra A is **exact** if whenever

$$0 \to B \to C \to D \to 0$$

is a short exact sequence of C^*-algebras, the sequence

$$0 \to B \otimes_\sigma A \to C \otimes_\sigma A \to D \otimes_\sigma A \to 0$$

is also exact.

It is known [3] that if A is exact and G is an amenable group, then $A \rtimes_\alpha G$ is an exact C^*-algebra. I would like to prove the analogous result for groupoid crossed products.

References

[1] Geoff Goehle, *Groupoid crossed products*, Ph.D. thesis, Dartmouth College, May 2009.

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[3] Eberhard Kirchberg, *Commutants of unitaries in UHF algebras and functorial properties of exactness*, Journal für die Reine und Angewandte Mathematik **452** (1994), 39–77.

[4] Aidan Sims and Dana P. Williams, *Amenability for Fell bundles over groupoids*, arXiv:1201.0792 [math.OA], January 2012.

[5] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, no. 134, American Mathematical Society, Providence, RI, 2007.