Efficient numerical solution of Helmholtz and electromagnetic scattering from periodic arrays of cylindrically symmetric objects

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Method of Fundamental Solution (MFS)

Introduction

MFS is an efficient way to solve partial differential equations (PDEs). The basic idea of MFS is to approximate the solution u by a linear combination of fundamental solutions of the problem:

$$u(\mathbf{r}) \approx \sum_{j=1}^{N} c_j \Phi(\mathbf{r} - \mathbf{r}'_j) \quad (1)$$

where N is the number of approximating functions each of which is centered at $\mathbf{r}'_j$ and have coefficient c_j.

Fundamental solution

The fundamental solution for Laplace and Helmholtz equation is the following:

Laplace

$$\begin{cases} \Phi(r) = -\frac{1}{2\pi}\ln r, & \text{n=2.} \\ \Phi(r) = \frac{1}{4\pi r}, & \text{n=3.} \end{cases}$$

Helmholtz

$$\begin{cases} \Phi(r) = \frac{i}{4}H_0^{(1)}(kr), & \text{n=2.} \\ \Phi(r) = \frac{e^{ikr}}{4\pi r}, & \text{n=3.} \end{cases}$$

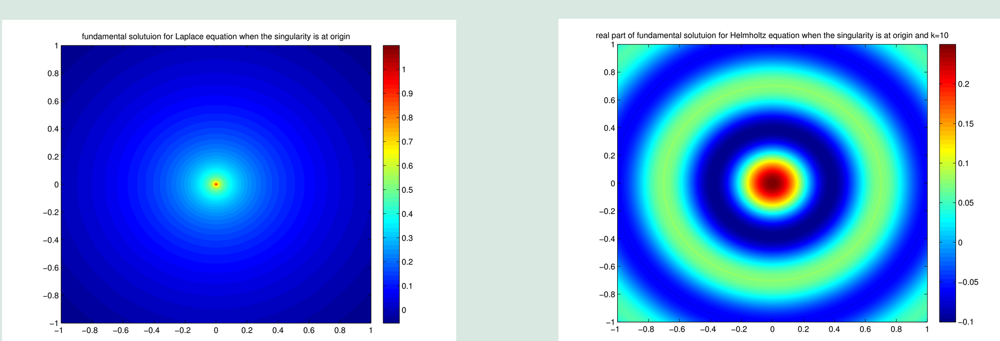

Figure 1: 2D fundamental solution centered at origin

Example

Now, we use a test function $u = e^x \cos(y) - e^y \sin(x)$ to demonstrate how to use MFS to solve a PDE. It is easy to verify that u satisfy Laplace equation.

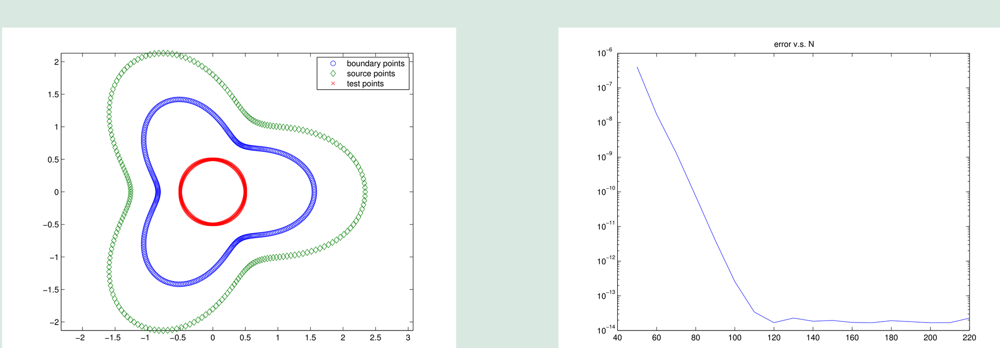

Figure 2: set up for 2D Laplace problem and exponential convergence

We match boundary conditions at M points, using N unknown coefficients. So equation (1) becomes

$$A\mathbf{c} = \mathbf{b} \quad (2)$$

where A is a known M-by-N matrix, and $\mathbf{b}$ involves the boundary data. By solving the linear system, we get the coefficients c_j and we can then evaluate the solution at test points and compare to the exact solution and get the error. Figure 2 (b) clearly shows that MFS is exponential convergence as N increases. This is in consistent with Barnett's and Katsurada's work [2][5][4].

Numerical Results

MFS combined with Fourier method

MFS applied in 3D can be computationally costly. But for cylindrically symmetric objects, we can exploit the symmetry and further simplify the problem:

$$K_n(\rho, z, \rho', z') = \int_0^{2\pi} e^{in\theta'} \Phi(\mathbf{r} - \mathbf{r}')d\theta' \approx \frac{2\pi}{P}\sum_{m=1}^{P} e^{in\frac{2\pi m}{P}} \Phi(\mathbf{r} - \mathbf{r}'_m) \qquad n = 1, 2, 3 \ldots P \quad (3)$$

where (ρ, z), (ρ', z') are the location of target point and MFS source ring in the ρz plane, respectively. P is the number of Fourier modes. $\mathbf{r} = (\rho, z, 0)$, $\mathbf{r}'_m = (\rho', z', \frac{2\pi m}{P})$ are the locations of the target point and m-th charge on the ring in a 3D cylindrical coordinate system, respectively. All the kernels can be evaluated once by the Fast Fourier Transform (FFT). The original 3D problem becomes P copys of independent 2D problems, one for each Fourier mode.

Helmholtz equation for high-frequency acoustics

We have implemented MFS for high-frequency Helmholtz problem. In figure 3(a), an acoustic wave is traveling in the $+y$ direction and transmitted into the object and was measured on the plane $y = 1.5$ (see figure 3a). We can clearly see the shadow of the object. This method gives the error of order $O(10^{-10})$ even with high frequency $k_+ = 50, k_- = 75$ (31 wavelengths in diameter) and it only takes 66 seconds to get the coefficients. Once the matrix is factorized, new incident waves can be solved in 2 seconds each. Figure 3b shows another wave traveling in the $-z$ direction, met the objects, transmitted it and measured its intensity at plane $z = -2$ with frequency $k_+ = 10, k_- = 30$.

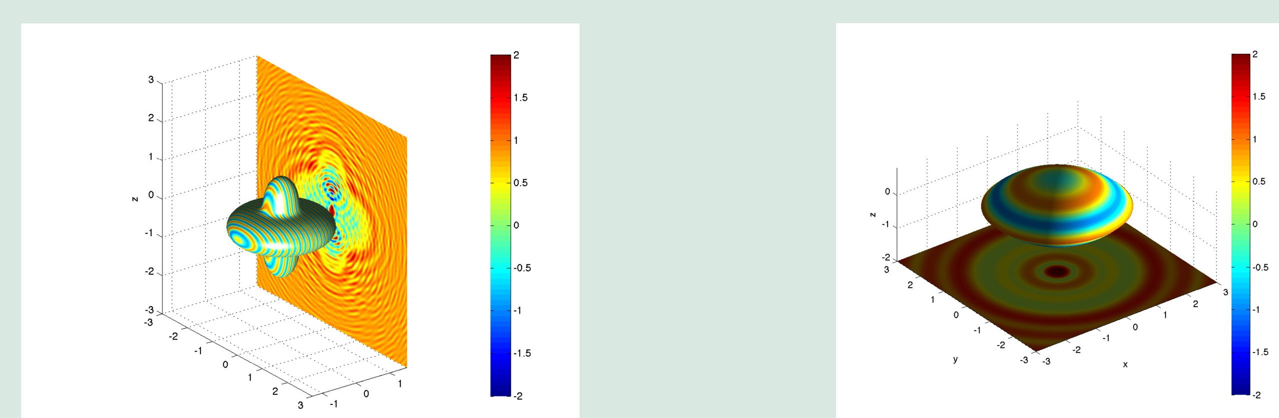

Figure 3: Helmholtz transmission problem

Full Maxwell equation

The fundamental solution to Maxwell equation is just the electric and magnetic field generated by a single electric or magnetic dipole. Here we take the electric dipole as the MFS source point. The electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$ at point $\mathbf{r}$ generated by a unit electric dipole located at $\mathbf{r_0}$ and oriented along τ is [1] ($\mathbf{R} = \mathbf{r} - \mathbf{r_0}$):

$$\mathbf{E} = \Big[\frac{3\mathbf{R}(\mathbf{R}\tau) - \tau\mathbf{R}^2}{R^5}(1 - ikR) - \frac{k^2\mathbf{R}\times(\mathbf{R}\times\tau)}{R^3}\Big]\frac{e^{ikR}}{4\pi\epsilon\epsilon_0} \qquad \mathbf{H} = \Big[\frac{1}{R^2} - \frac{ik}{R}\Big](\mathbf{R}\times\tau)\frac{ik_0 e^{ikR}}{4\pi R(\mu_0\epsilon_0)^{1/2}\mu} \quad (4)$$

Figure 1d plots the z-component of the electric field E_z in the xy plane. The boundary in this problem is a unit sphere. We can clearly see that the E_z is continuous across the boundary, which indicates the accuracy of the solution. The error tested in the figure 4(d) case is in the order $O(10^{-12})$ and it takes 63 seconds to get the coefficients. Once the matrix is factorized, new incident waves can be solved in 0.63 seconds each.

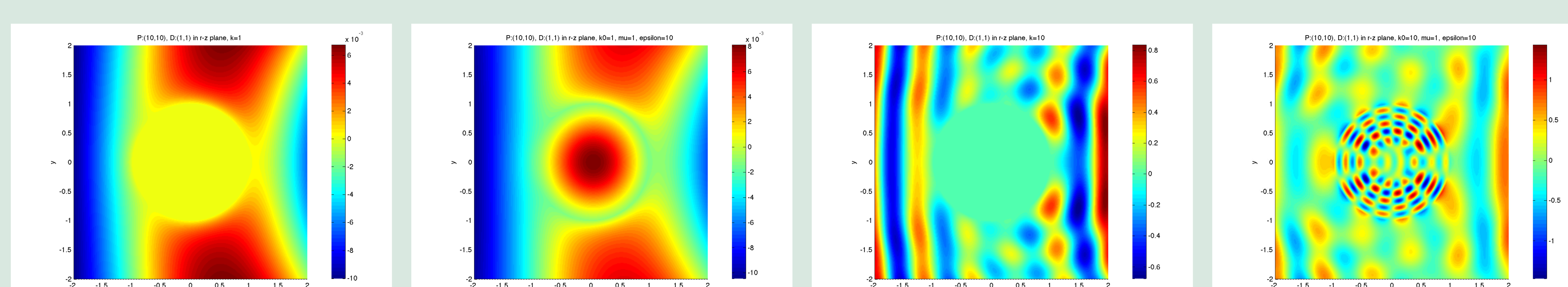

Figure 4: Plots of E_z in the plane $z = 0$, with dipole source locating at (10, 10, 0) and orienting at (1, 1, 0): (a) $k_0 = 1$ and the domain is perfect electric conductor; (b)$k_0 = 1$ and the domain is dielectrics; (c) $k_0 = 10$ and the domain is perfect electric conductor; (d)$k_0 = 10$ and the domain is dielectrics.

Future Work

Periodization

MFS can also be used to deal with periodization. In the example shown in figure 5, there is a 2D square box with length 2 and centered at origin. We can add MFS source points outside of the box and impose the periodic conditions to get the unknown charge strengths of the MFS points. Figure 5 shows the plot of the solution inside of the box. We can clearly see that the solution satisfies the periodic condition. The error tested in this case is in the order $O(10^{-13})$ and it takes 0.06 seconds to get the coefficients. We will complete the periodization in 3D.

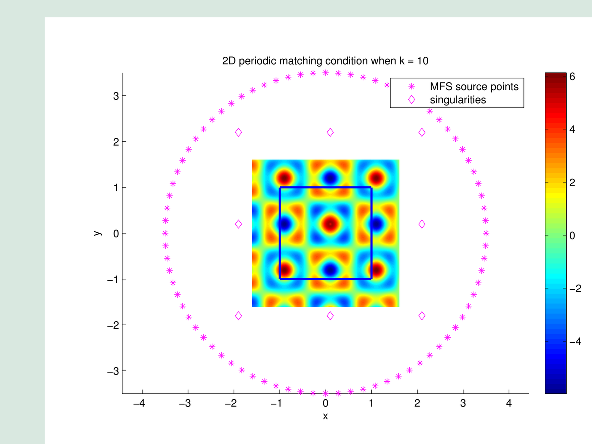

Figure 5: 2D periodic problem, where the blue box is the periodic unit

Periodic arrays of axisymmetric objects

The next goal is to solve the problem involving an array of axisymmetric objects (see figure 6). This requires to combine two kinds of MFS source points. One is the MFS source points that model the fundamental solution of Helmholtz equation or Maxwell equation. Another is the MFS source points that lie outside of the box to model the imposed periodic condition. We plan to compute it using a 3D generalization of recent method of Barnett–Greengard [3].

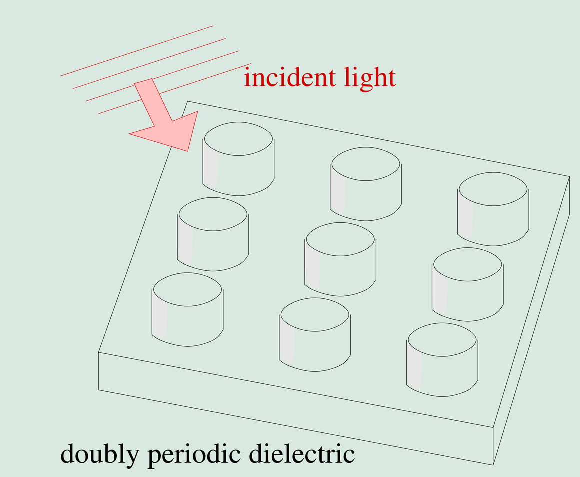

Figure 6: wave scattering from periodic arrays of axisymmetric objects

References

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