Pattern Occurrences in Random Walks

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Patterns in Economics

In recent years, the concept of "forbidden patterns" has been applied to financial time series (the historical prices in the economic market) in order to determine whether the behavior of prices is closer to a deterministic or random process. The idea of a pattern in a time series can loosely be described as the shape of the graph over a finite set of consecutive days - we concern ourselves with the relative ordering of the prices at each day and represent a pattern with a permutation on the set $1, 2, 3, \ldots, n$.

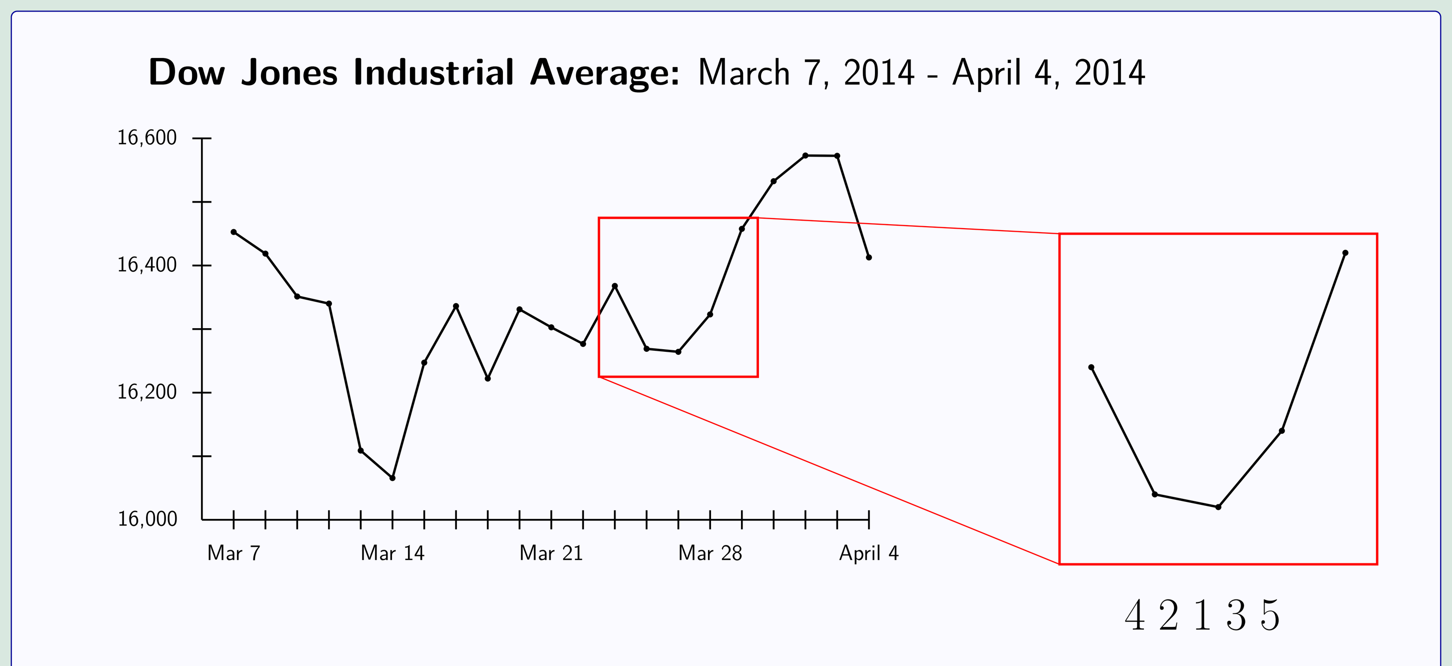


Figure 1 : The occurrence of the pattern 42135 in the Dow Jones Industrial Average

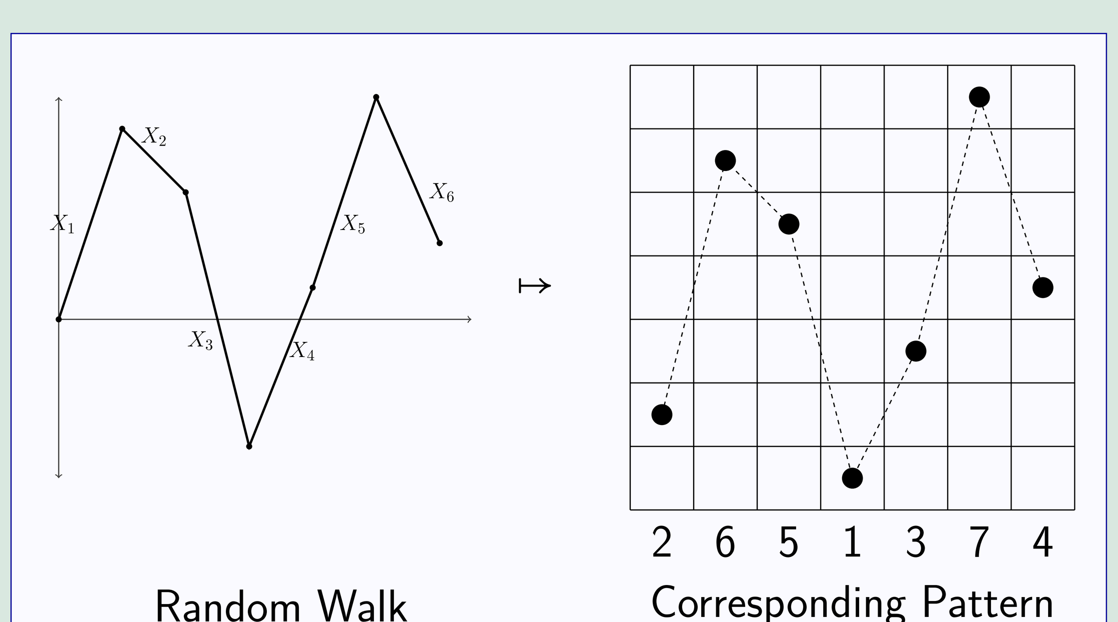
Over time, one can examine the patterns that occur and do not occur. Those that do not occur are called the "forbidden patterns." The occurrence of forbidden patterns can be indicators that one is dealing with a deterministic sequence.

The goal of this project is to further examine the behavior of patterns in a random time series.

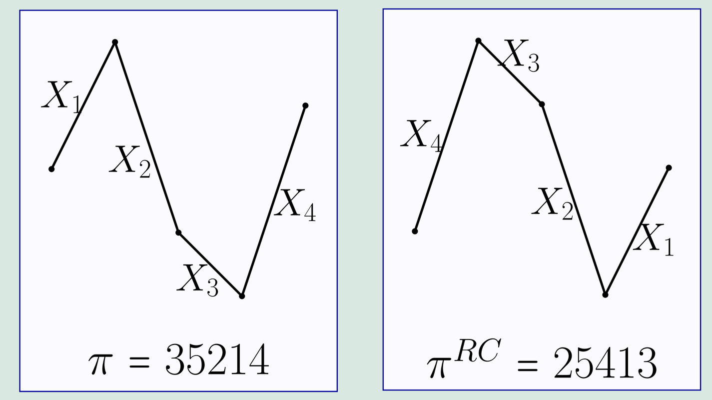
Random Walks and Patterns

Constructing a random walk:

- Decide on your probability distribution
- Choose $n-1$ independent and identically distributed random variables, $X_1, X_2, X_3, \ldots, X_{n-1}$
- The X_i's are the step sizes and define the random walk.
- The relative ordering of the vertices determines the corresponding pattern.



The Reverse-Complement Phenomenon



- π^{RC} is the rotation of π by $180°$.
- π and π^{RC} will have equal probability of occurring regardless of the probability distribution!

The Big Question: For any pattern, π, what other patterns have the same probability of occurring, regardless of the probability distribution?

Bordered Cylindrical Blocks

Examples of bordered and unbordered cylindrical blocks in the pattern $8, 5, 7, 6, 4, 9, 2, 10, 1, 3$.

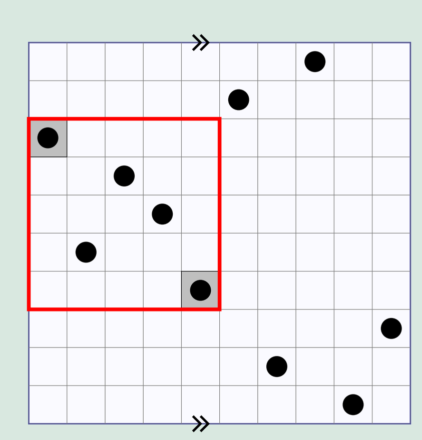


Figure 2 : The bordered cylindrical block $8, 5, 7, 6, 4$

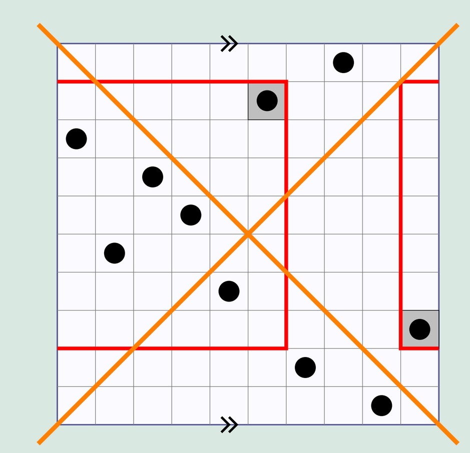


Figure 4 : This block is bordered, but it is **not** cylindrical!

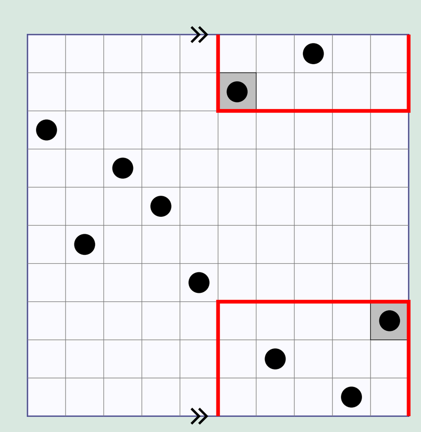


Figure 3 : The bordered cylindrical block $9, 2, 10, 1, 3$

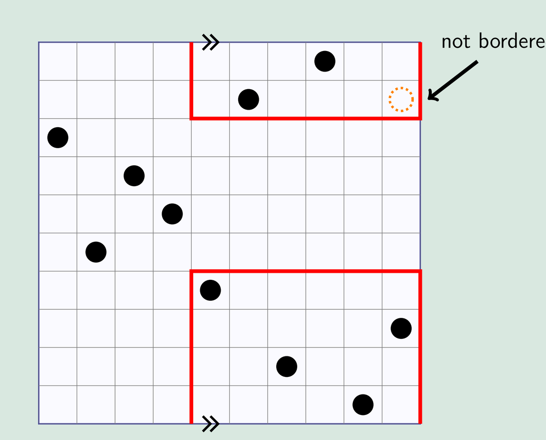
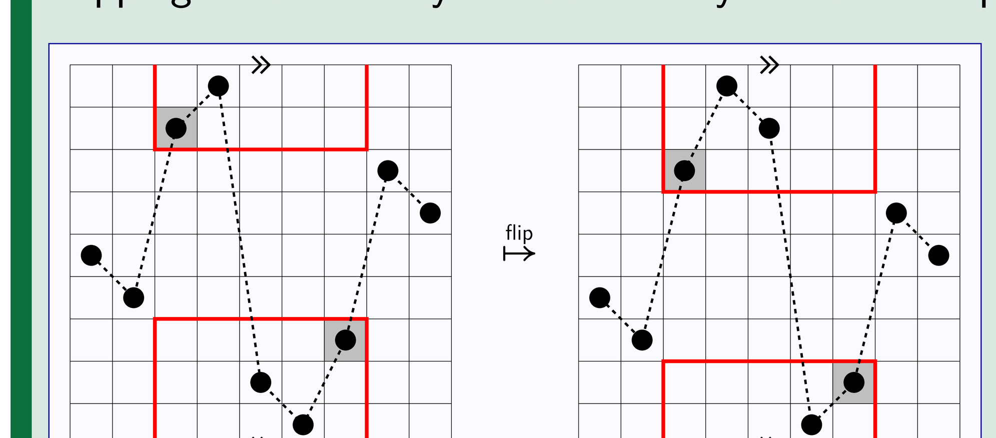


Figure 5 : The unbordered cylindrical block $4, 9, 2, 10, 1, 3$

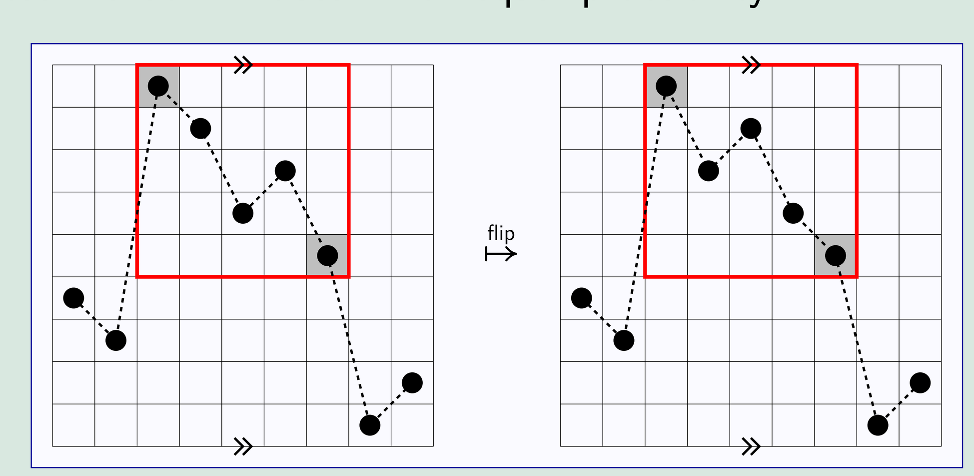
- A **cylindrical block** is a block in a pattern embedded on a cylinder (associate the top and bottom edges of the grid).
- A **bordered block** is a block that has entries in opposite corners.

Flipping Bordered Cylindrical Blocks

Flipping a bordered cylindrical block yields another pattern that will occur with equal probability.



548921376 has equal probability of occurring as 437981265



439867512 has equal probability of occurring as 439786512

Canonical Inequalities

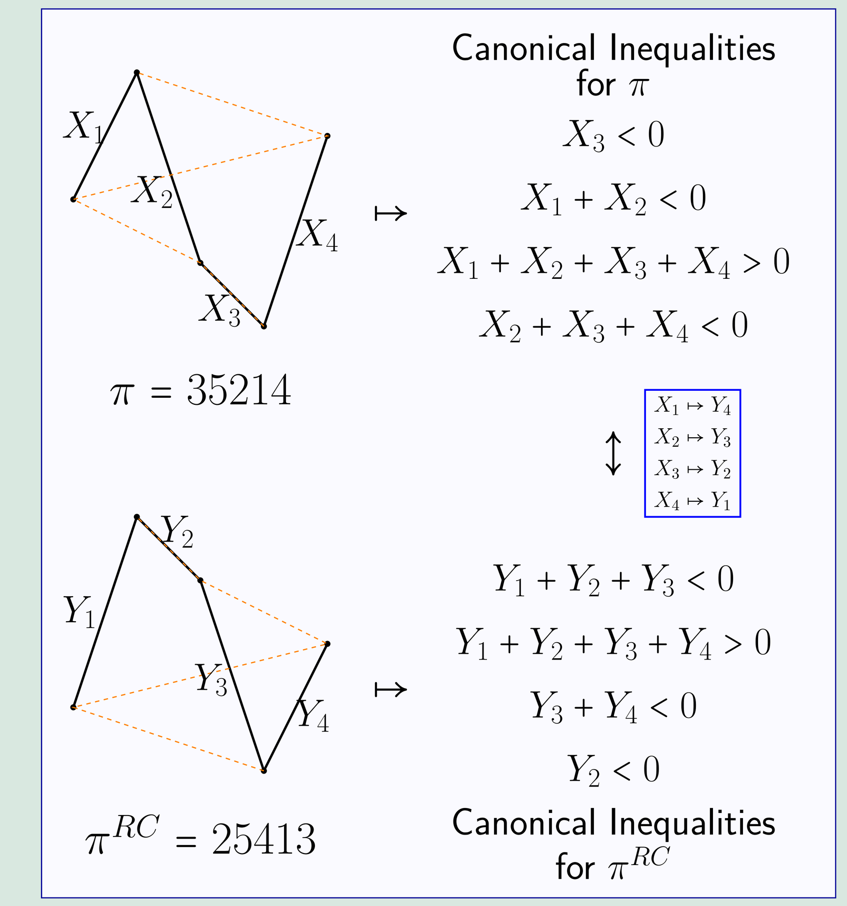
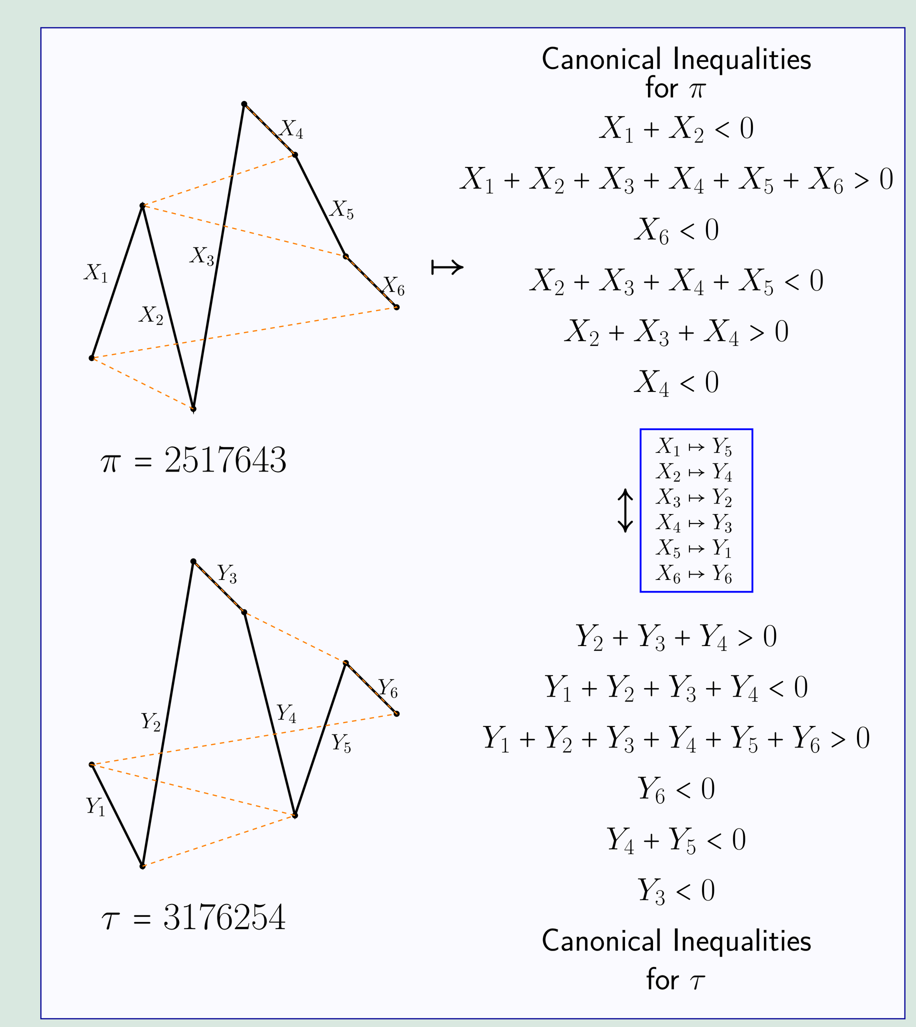


Figure 6 : π and π^{RC} have the same inequalities up to bijection

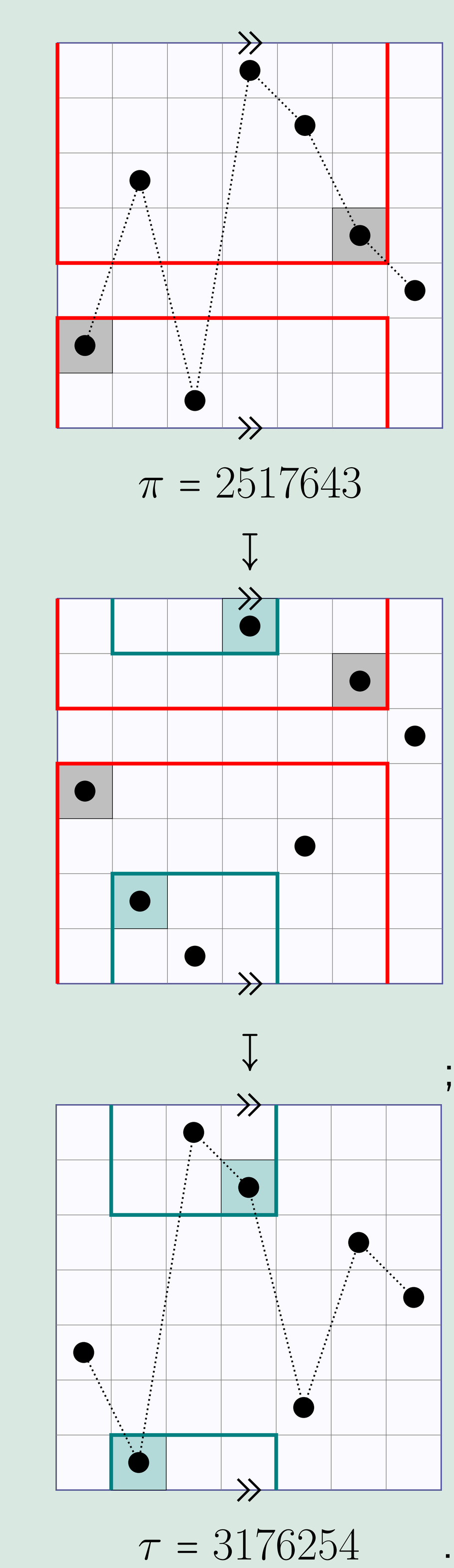
- Encode relationships of random variables $X_1, X_2, \ldots, X_{n-1}$ in "canonical" inequalities.
- Two permutations have the same canonical inequalities up to bijection if there is a bijection that maps the random variables of one permutation to the other.

Main Result

Theorem. Two patterns have the same canonical inequalities up to bijection if and only if one can be obtained from the other by a sequence of flips of bordered cylindrical blocks and the reverse-complement operation.



In this example, π and τ have the same canonical inequalities up to bijection, and we can find the sequence of flips of bordered cylindrical blocks that maps π to τ.



References
[1] M. Zanin, *Forbidden Patterns in Financial Time Series.* Chaos 18 (2008), 013119.