Chapter 16

Journey into Space

We all live in three spacial dimensions, and are attuned to thinking three-dimensionally even though we see, draw and paint, for the most part, only in two. Thus, geometrical problems in three dimensions, while potentially much harder than problems in the plane, can appeal to our intuition in pleasant and even useful ways.

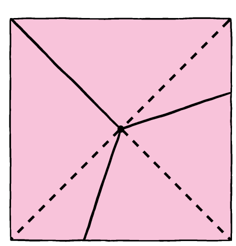
Some of the puzzles that follow are set naturally in three dimensions; others can be placed there to good effect.

Easy Cake Division

Can you cut a cubical cake, iced on top and on the sides, into three pieces each of which contains the same amount of cake and the same amount of icing?

Solution:

You can get any number k of pieces with the same amount of cake and the same amount of icing in the following neat way: Looking at the cake from above, divide the perimeter of the square top of the cake into k equal parts and make straight, vertical cuts to the middle of the square from each boundary point on the perimeter.



That works because each piece, looked at from the top, is the union of triangles with the same altitude (namely, half the side of the cake) and the same total base (namely, $1/k$ times the perimeter), as in the figure below.

Speaking of cubes, here's a question about painting them.

Painting the Cubes

Can you paint 1,000 unit cubes with 10 different colors in such a way that for any of the 10 colors, the cubelets can be arranged into a $10 \times 10 \times 10$ cube with only that color showing?

Solution:

Note that the numbers work out perfectly: 6000 faces need to be colored, and the number of faces showing on the big cube is $6 \times 100 = 600$. Thus we'll need to paint exactly 600 faces with each color, and when we put the big cube together, we can't afford to bury even a single face of the designated color inside.

It turns out that despite these restrictions, there are many ways to color the unit cubes that will do the trick. But finding one by hammer and tongs is a pain in the neck.

Instead, number the colors 0 through 9. Apply paint to all the cubes in the infinite 3-dimensional cubic grid, as follows: Paint both sides of all the faces on the plane $x = 0$ with color 0, all faces on $x = 1$ with color 1, etc., rotating back to color 0 for the plane $x = 10$, and so forth. Do the same for the y-planes and z-planes.

This scheme colors every face of every unit cube with a color corresponding to the rightmost digit of that face's u-coordinate, where u is x, y or z according to which axis is perpendicular to the face. In particular, the scheme colors unit cubes in 10^3 different ways. But now, if you want your big cube to display color i on its outside, cut out your big cube from space via planes $x = i$, $x = i+10$, $y = i$, $y = i+10$, $z = i$, and $z = i+10$. This will use each type of unit cube once, and you are done! ♡

Of course, you can replace 10 by any positive integer n (that is, you can color n^3 unit cubes with which you can then make an $n \times n \times n$ cube with any given color the only one showing). The above argument still works, and in fact versions of it work in other dimensions, as well.

Just don't think too hard about what it means to "paint" a 3-dimensional face of a 4-dimensional hypercube.

Curves on Potatoes

Given two potatoes, can you draw a closed curve on the surface of each so that the two curves are identical as curves in three-dimensional space?

Solution:

Yes, you can: Intersect the potatoes!

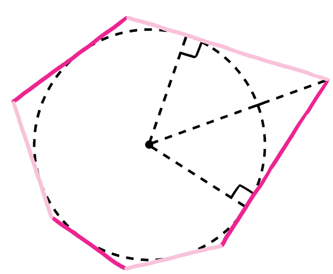
Well, maybe a few more words of explanation would not go amiss. Think of the potatoes as holograms and bring them together so that they overlap in space. Then the intersection of their surfaces will describe a curve (possibly more than one curve) that lives on both surfaces, and solves the puzzle.

Painting the Polyhedron

Let P be a polyhedron with red and green faces such that every red face is surrounded by green ones, but the total red area exceeds the total green area. Prove that you can't inscribe a sphere in P.

Solution:

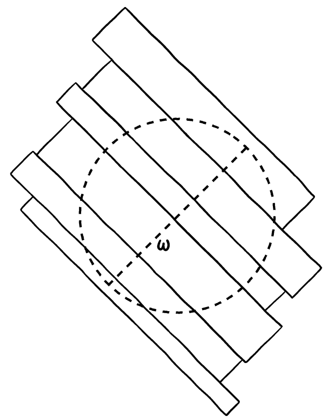
Assume the sphere is inscribed, and triangulate the faces of P using the points where the sphere is tangent. Then the triangles on either side of any edge of P are congruent, thus have the same area; at most, one of every such pair of triangles is red. It follows that the red area is at most equal to the green area, contradicting our assumptions. ♡



The illustration shows a two-dimensional version, where sides and vertices of a polygon take the place of faces and edges of P.

Boarding the Manhole

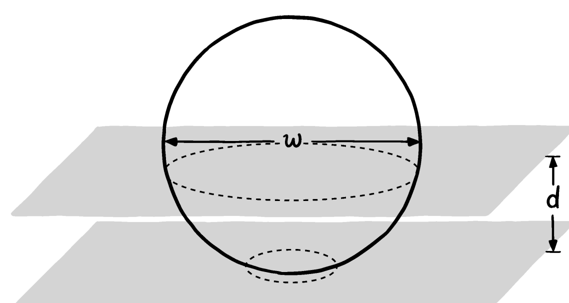
An open manhole 4 meters in diameter has to be covered by boards of total width w. The boards are each more than 4 meters in length, so if $w \geq 4$ it's obvious that you can cover the manhole by laying the boards side by side (see figure below). If w is only 3.9, say, you still have plenty of wood and you are allowed to overlap the boards (but not cut them up) if you want. Can you still cover the manhole?



Solution:

Intuitively speaking, even though you have plenty of wood, it seems that you have to waste a lot of it either by covering the center of the manhole many times, or laying boards too close to the edge to be of much use. Putting it another way, it's relatively expensive to cover parts of the manhole far from the center. Can you quantify that statement in a useful way?

It turns out you can, using a famous fact known sometimes as Archimedes' hat box theorem. Archimedes used his "method of exhaustion" (these days, we would use calculus) to show that if a sphere of radius r is intersected by two parallel planes a distance d apart, as in the figure below, the surface area of the sphere between the planes is $2\pi rd$. In particular, it doesn't depend on where the planes cut the sphere, but only on how far apart the planes are. (The "hat box" presumably alludes to a cylinder of radius r perpendicular to the planes; the cylinder's area between the planes is that same $2\pi rd$.)



Where do we get our planes from? Let's replace the manhole by a sphere, also of diameter w, whose equator is the circumference of the manhole. Each board is replaced by a vertical *slab* consisting of two parallel vertical planes whose distance d apart is the width of the board. Now the key observation: Arranging the boards to cover the manhole is equivalent to arranging the slabs to cover the sphere.

But by Archimedes' theorem, the slabs cannot cover more than $2\pi \frac{w}{2} d = \pi w d$ of the surface of the sphere, which has area πw^2. So if $d < w$, you're stuck. ♡

Now we are set up perfectly to conquer the next puzzle.

Slabs in 3-Space

A "slab" is the region between two parallel planes in three-dimensional space. Prove that you cannot cover all of 3-space with a set of slabs the sum of whose thicknesses is finite.

Solution:

In a way the conclusion seems obvious; if you can't cover space when the slabs are parallel and disjoint, *surely* you can't cover when they waste space by overlapping. But infinity is a tricky concept. The slabs all have infinite volume, so what's to prevent them from covering anything they want?

Actually, we know from the previous puzzle that they have trouble covering a large ball. If the total thickness of the slabs is T, then, by Archimedes' theorem, they can't cover the surface of a ball of diameter greater than T, thus they can't cover the ball itself.

Thus they certainly can't cover all of 3-space. But it's odd that to show the latter, we seem to need to reduce the problem to just a finite piece of space.

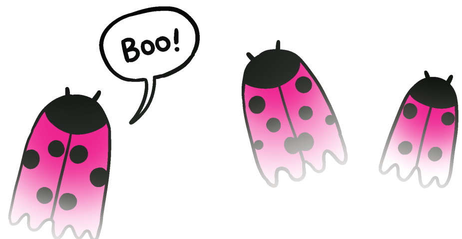
Bugs on Four Lines

You are given four lines in a plane in general position (no two parallel, no three intersecting in a common point). On each line a ghost bug crawls at some constant velocity (possibly different for each bug). Being ghosts, if two bugs happen to cross paths they just continue crawling through each other uninterrupted.

Suppose that five of the possible six meetings actually happen. Prove that the sixth does as well.

Solution:

As the title of this chapter suggests, lifting the puzzle into space provides an elegant solution. How? By means of a time axis. Suppose every pair of bugs meets except bug 3 with bug 4. Construct a time axis perpendicular to the plane of the bugs, and let g_i be the graph (this time, in the sense of graphing a function) of the i^{th} bug in space. Since each bug travels at constant speed, each such graph is a straight line; its projection onto the plane of the bugs is the line that bug travels on. If (and only if) two bugs meet, their graphs intersect.



The lines g_1, g_2 and g_3 are coplanar since all three pairs intersect, and the same applies to g_1, g_2 and g_4. Hence all four graphs are coplanar. Now g_3 and g_4 are certainly not parallel, since their projections onto the original plane intersect, thus they intersect on the new common plane. So bugs 3 and 4 meet as well.

Circular Shadows II

Show that if the projections of a solid body onto two planes are perfect disks, then the two projections have the same radius.

Solution:

The conclusion seems eminently reasonable, yet it's not completely obvious how to prove it.

An easy way to make your intuition rigorous is to select a plane which is simultaneously perpendicular to the two projection planes, and move parallel copies of it toward the body from each side. They hit the body at the opposite edges of each projection, so that the distance between the parallel planes at that moment is the common diameter of the two projected circles. ♡

Box in a Box

Suppose the cost of shipping a rectangular box is given by the sum of its length, width, and height. Might it be possible to save money by fitting your box into a cheaper box?

Solution:

The issue here is that if packed diagonally, the inside box could have some dimensions that exceed the greatest dimension of the outside box. For example, a narrow needle-like box of length almost $\sqrt{3}$ could be packed inside a unit cube. This particular example wouldn't provide a way to cheat, because in this case the total cost of the outside cube would be 3, much more than the 1.7 or so you might have to pay for the inside box. But how do we know that some better example won't turn up?

Here's a delightful argument that lets ε go to infinity(!) in order to show that you can't cheat the system.

Suppose your box is $a \times b \times c$, and let R stand for the region of space occupied by your box including its interior. Suppose R can be packed into an $a' \times b' \times c'$ box R' with $a + b + c > a' + b' + c'$. Let $\varepsilon > 0$ and consider the region R_ε consisting of all points in space that are within distance ε of R. This R_ε region will be a rounded convex shape containing your box R. The volume of R_ε is equal to your box's volume plus $2\varepsilon(ab + bc + ac)$ for the volume added to the sides, plus $\pi\varepsilon^2(a + b + c)$ for the volume added to the edges, plus $\frac{4}{3}\pi\varepsilon^3$ for the volume (consisting of the eight octants of a ball) added to the corners. If you do the same to the outside box R' you get a region R'_ε which of course must contain R_ε. But that's impossible! In the expression for $\mathrm{vol}(R'_\varepsilon) - \mathrm{vol}(R_\varepsilon)$, the ε^3 terms cancel, so if you take ε to be large, the dominant term is the ε^2 term, namely $\pi\varepsilon^2((a' + b' + c') - (a + b + c))$, which is negative. ♡

Angles in Space

Prove that among any set of more than 2^n points in $\mathbb{R}^n$, there are three that determine an obtuse angle.

Solution:

It makes sense that the 2^n corners of a hypercube represent the most points you can have in n-space without an obtuse angle. But how to prove it?

Let $x_1, \ldots, , x_k$ be distinct points (vectors) in $\mathbb{R}^n$, and let P be their convex closure. We may assume P has volume 1 by reducing the dimension of the space to the dimension of P, then scaling appropriately; we may also assume x_1 is the origin (i.e., the 0 vector). If there are no obtuse angles among the points, then we claim that for each $i > 1$, the interior of the translate $P + x_i$ is disjoint from the interior of P; this is because the plane through x_i perpendicular to the vector x_i separates the two polytopes.

Furthermore, the interiors of $P + x_i$ and $P + x_j$, for $i \neq j$, are disjoint as well; this time via the separating plane running through $x_i + x_j$ perpendicular to the vector $x_j - x_i$. We conclude that the volume of the union of the $P + x_i$, for $1 \leq i \leq k$, is k.

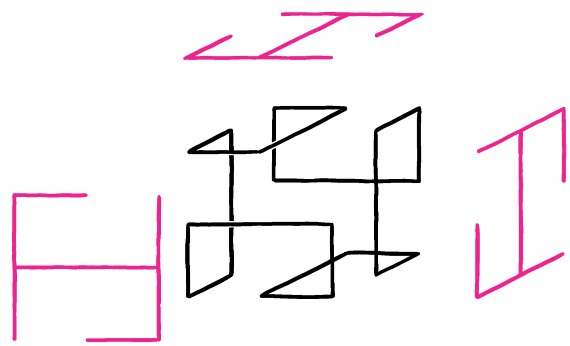
However, all these polytopes lie inside the doubled polytope $2P = P + P$, whose volume is 2^n. Hence, $k \leq 2^n$ as claimed!

Curve and Three Shadows

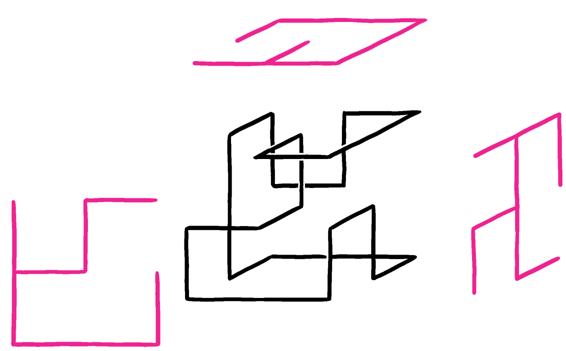
Is there a simple closed curve in 3-space, all three of whose projections onto the coordinate planes are trees? This means that the shadows of the curve, from the three coordinate directions, may not contain any loops.

Solution:

No one knew the answer to this question until such a curve was actually found, by John Terrell Rickard of Advanced Telecommunication Modules Ltd. in Cambridge, UK. His beautiful, symmetrical solution is shown below with its three shadows. I have no idea how Rickard found it.



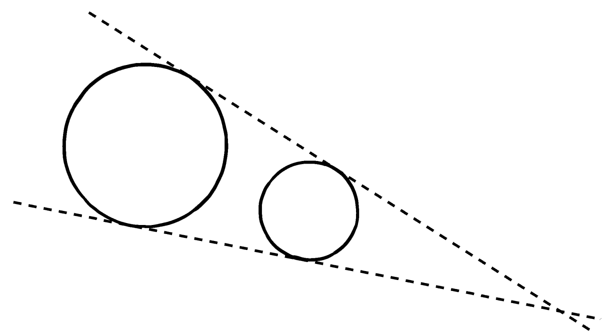
Later, Donald Knuth, author of the multi-volume classic *The Art of Computer Programming*, programmed a computer to see if there were any other solutions that, like Rickard's, could be carved from a $2 \times 2 \times 2$ region of a cubical grid. In the time it took to push his *enter* button (according to Knuth), the computer found that there was just one other solution, shown below. Knuth's solution, in contrast to Rickard's, has no symmetry at all.



If you found either one of these curves, more power to you!

Our theorem is perhaps the world's best-known example of proving a theorem about plane geometry by moving into 3-dimensional space. But there's an ironic twist.

Given any two circles in the plane of different radii, neither contained in the other, we can construct two lines that are tangent to the circles such that for each line, the circles are on the same side of the line. (See picture below). These lines will intersect at some point on the plane, known as the *Monge point* of the two circles.



Theorem. *Suppose three circles are given, all of different radii, none contained in another. Then the three Monge points determined by the three pairs of circles all lie on a line.*

This is what the picture looks like (we have taken the circles to be disjoint, but in fact they're allowed to overlap).

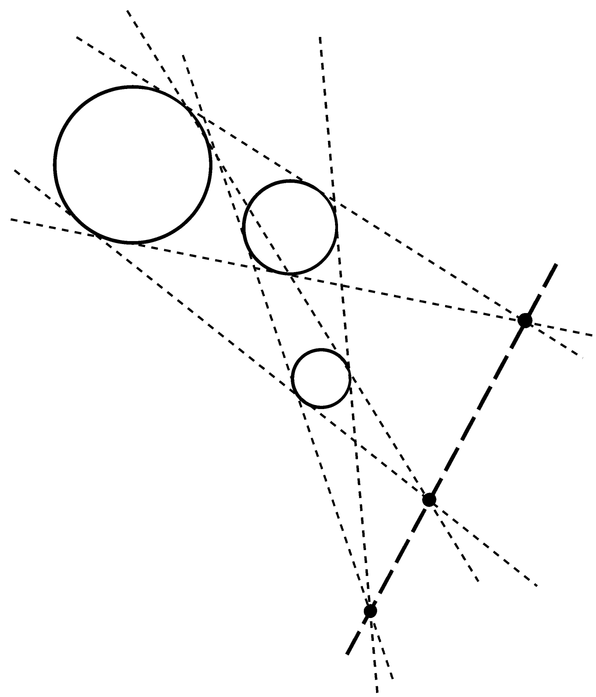
The theorem is attributed to Gaspard Monge, 1746–1818, a distinguished French mathematician and engineer. On Wikipedia (as I write this) you will find a proof which is elegant, famous, and wrong! Here's how it goes:

Replace each circle by a sphere whose equator is that circle, so now you have three spheres (of different diameters) intersecting the original plane at their equators. Pick two of them: Instead of two tangent lines you now have a whole tangent *cone* whose apex is the Monge point of the circles.

Now, take a plane tangent to the three spheres, thus also tangent to the three cones. The three Monge points will all lie on that plane, as well as on the original plane. But the two planes intersect in a line, so the Monge points are all on that line, QED!

It's not easy to spot the hole in this proof. The difficulty is that there may not *be* a plane tangent to the three spheres; for example, if two of the circles are large and the third is between them and smaller.

The following proof, attributed by cut-the-knot.org to Nathan Bowler of Trinity College, Cambridge, works by erecting *cones* instead of spheres on top of the circles. Call them C_1, C_2 and C_3, and let them all be "right" cones—that is, they support $90°$ angles at their apices. (Actually, we only need all the cones to have the same angle.) Each pair of cones determines two (outside) tangent planes, say P_1 and Q_1 (for cones C_2 and C_3), P_2 and Q_2 (for cones C_1 and C_3), and finally P_3 and Q_3 (for cones C_1 and C_2).



Each pair of planes P_i, Q_i intersects in a line L_i which passes through the apex of both tangent cones, as well as through the point where the corresponding circle tangents meet. Thus, in particular, L_1 and L_2 meet at the apex of C_3, L_1 and L_3 at the apex of C_2, and L_2 and L_3 at the apex of C_1. Hence the three lines of intersection are coplanar (all lie on the plane determined by the three apices); the intersection of that plane with the original plane of the circles is a line through the three foci, and this time we are really done. ♡

There are other ways to prove Monge's theorem and as you might suspect, one of them is to transform the plane in a circle-preserving way so as to escape annoying initial configurations that screw up the spheres proof. But it's curious, and a bit sobering, that the unfixed sphere argument lives on as the "standard" proof of Monge's theorem.