Chapter 7

The Law of Small Numbers

It sounds silly, but many otherwise-very-smart people, given a puzzle with numbers in it, treat those numbers as if underlined in a sacred text. This is math—you're allowed to change the numbers and see what happens! If the numbers in a puzzle are dauntingly large, replace them with small ones. How small? As small as possible, without making the puzzle trivial; if that doesn't give you enough insight, make them gradually bigger.

Domino Task

An 8×8 chessboard is tiled arbitrarily with 32 2×1 dominoes. A new square is added to the right-hand side of the board, making the top row length 9.

At any time you may move a domino from its current position to a new one, provided that after the domino is lifted, there are two adjacent empty squares to receive it.

Can you retile the augmented board so that all the dominoes are horizontal?

Solution:

Yes. Let T be the "snake" tiling obtained by placing four vertical tiles on the left column, three on the right column (missing the top and bottom squares), and filling in all but the lower right square with horizontal dominoes. Our goal is to create this tiling and then shift it to get the desired horizontal tiling of the chessboard.

We construct the snake tiling from the top. Since the 32 dominoes cover 64 of the 65 squares of our extended board, there is always one uncovered square, which we call the "hole." Suppose the dominoes covering the remaining squares in the hole's row are not all horizontal; then we can move the vertical tile nearest the hole on either side to a horizontal position after shifting some horizontal dominoes. Let's call this process "flattening."

Since flattening increases the number of horizontal dominoes, it must eventually terminate with the hole on a row containing only horizontal dominoes.

But that must be the top row, since only the top row has an odd number of squares.

By shifting the top-row dominoes and filling the top-left square with a vertical domino, we create the top row of the snake. Now we return to flattening (but not touching the vertical tile at the upper left); this winds up with the hole on the second row, and now we can shift to make that row match the snake.

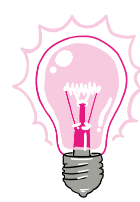
We repeat until the whole snake is created; flattening the snake concludes the whole process. How to find this curious algorithm? Try the problem first on a 4×4 board!

Spinning Switches

Four identical, unlabeled switches are wired in series to a light bulb. The switches are simple buttons whose state cannot be directly observed, but can be changed by pushing; they are mounted on the corners of a rotatable square. At any point, you may push, simultaneously, any subset of the buttons, but then an adversary spins the square. Is there an algorithm that will enable you to turn on the bulb in at most a fixed number of spins?

Solution:

Looking at a simpler version of this puzzle is crucial. Consider the two-switch version, where all you've got are two buttons on diagonally opposite corners of the square. Pushing both buttons will ascertain whether the two switches were both in the same state, since then the bulb will light (if it wasn't already lit). Otherwise, push one button, after which they *will* be in the same state, and at worst one more operation of pushing both buttons will turn on the bulb. So three operations suffice.



Back to the four-switch case. Name the buttons N, E, S, and W after the compass directions, although of course the botton you're calling N now might be the button you will call E, W or S after a spin. Suppose that at the start, diagonally opposite switches (N and S, E and W) are in the same state—both on or both off. Then you can treat opposite pairs as a single button and use the two-button solution: push both pairs (i.e., all four switches); then one pair (which may as well be N-S); then both pairs again, and you're done. So begin with those three operations; if the light doesn't go on, then one or both of the opposite pairs must have been mismatched. Try flipping two neighboring switches, say N and E, then going back through your three-move two-button solution. Then you're fine if both pairs were mismatched. If not, push just one button; that'll either make both opposite pairs match, or both mismatch. Run through the two-button solution a third time. If the bulb is still off, push N and E again and now you *know* both opposite pairs match, and a fourth application of the two-button solution will get that bulb turned on.

In conclusion, pushing buttons NESW, NS, NESW, NE, NESW, NS, NESW, N, NESW, NS, NESW, NE, NESW, NS, NESW, will at some point turn the light on—fifteen operations. No sequence of fewer than 15 operations can be guaranteed to work because there are $2^4 = 16$ possible states for the four switches, and they all must be tested; you get to test one state (the starting state) for free.

Seeing the solution for four buttons, you can generalize to the case where the number of buttons is any power of two; if there are 2^k buttons, the solution will take, and require, $2^{2^k}-1$ steps. (When there are n buttons, they are located at the corners of a spinnable regular n-gon.)

The puzzle is insoluble when the number of buttons, n, is not a power of 2. Let's just prove that for three buttons, no fixed number of operations can guarantee to get that bulb on. (For general n, write n as $m \cdot 2^k$ for some odd number $m > 1$; it is m which plays the role of 3 in what follows.)

You may as well assume the switches are spun before you even make your first move. Suppose that before they are spun, the switches are not all in the same state. Then it is easy to check that *no matter what move you planned*, if you were unlucky with the spin, then after the spin and your move, the switches will still not all be in the same state.

It follows that you can never be sure that you have ever had all the switches in the same state, so no fixed sequence of moves can guarantee to light the bulb. It's curious that you can solve the problem for 32 buttons (albeit in about 136 years, at one second per operation), but not for just three buttons.

Candles on a Cake

It's Joanna's 18th birthday and her cake is cylindrical with 18 candles on its 18″ circumference. The length of any arc (in inches) between two candles is greater than the number of candles on the arc, excluding the candles at the ends.

Prove that Joanna's cake can be cut into 18 equal wedges with a candle on each piece.

Solution:

The conditions give some assurance that the candles are fairly evenly spaced; one way to say that is that as we move around the circumference from some fixed origin 0, the number of candles we encounter is not far from the distance we have traveled. Accordingly, let a_i be the arc-distance from 0 to the ith candle, numbered counterclockwise, and let $d_i = a_i - i$.

We claim that for any i and j, d_i and d_j differ by less than 1. We may assume $i < j$; suppose, for instance, that $d_j - d_i \leq -1$. Then $j-i-1 \geq a_j - a_i$, but j–i–1 is the number of candles between i and j, contradicting the condition. Similarly, if $d_j - d_i \geq 1$, then $d_i - d_j \leq -1$ and the same argument applies to the other arc, counterclockwise from j to i.

So the "discrepancies" d_i all lie in some interval of length less than 1. Let d_k be the smallest of these, and let ε be some number between 0 and d_k, so that all the d_i's lie strictly between ε and $1+\varepsilon$. Now cutting the cake at ε, ε+1, etc. gives the desired result.

How are you supposed to find this proof? By trying two candles, then three, instead of eighteen.

Lost Boarding Pass

One hundred people line up to board a full jetliner, but the first has lost his boarding pass and takes a random seat instead. Each subsequent passenger takes his or her assigned seat if available, otherwise a random unoccupied seat.

What is the probability that the last passenger to board finds his seat unoccupied?

Solution:

This is a daunting problem if you insist on working out what happens with 100 passengers; the number of possiblities is astronomical. So let's reduce the number to something manageable. With two passengers it's obvious that the probability that the second (i.e., last) get her own seat is 1/2. What about three passengers?

It's useful to number the seats according to who was supposed to sit there. If passenger 1 sits in seat 1, his assigned seat, then everyone will be get his or her own seat. If he sits in seat 3, then passenger 2 will get seat 2 and passenger 3 will get seat 1. Finally, if passenger 1 sits in seat 2, then whether passenger 3 gets seat 3 will depend on whether passenger 2 chooses seat 1 or seat 3. Altogether, the probability that passenger 3 gets her own seat is $\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2}$.

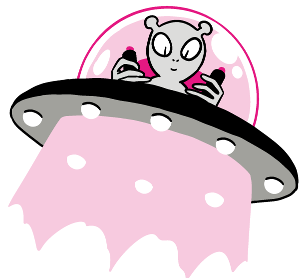
Interesting! Is it possible that the answer is always 1/2?

We notice that in the above analysis, the last passenger never ends up in seat 2. In fact, now that we think of it, we see that with n passengers total, the last passenger never ends up in seat i for $1 < i < n$. Why? Because when passenger i came on board, either seat i was already taken, or it is taken now. Thus, seat i will *never* be available to the last passenger. The only seats that passenger n could end up in are seat 1 and seat n.

We can't yet conclude that the probability that passenger n gets seat n is 1/2—we still need to argue that seat 1 and seat n are equally likely to be available at the end. But that's easy, because every time someone took a random seat, they were equally likely to choose seat 1 or seat n. Putting it another way, seat 1 and seat n were treated identically throughout the process; thus, by symmetry, each has the same likelihood of being open when passenger n finally gets on board.

Flying Saucers

A fleet of saucers from planet Xylofon has been sent to bring back the inhabitants of a certain randomly-selected house, for exhibition in the Xylofon Xoo. The house happens to contain five men and eightt women, to be beamed up randomly one at a time.



Owing to the Xylofonians' strict sex separation policy, a saucer cannot bring back earthlings of both sexes. Thus, it beams people up until it gets a member of a second sex, at which point that one is beamed back down and the saucer takes off with whatever it has left. Another saucer then starts beaming people up, following the same rule, and so forth.

What is the probability that the last person beamed up is a woman?

Solution:

Let's try some smaller numbers and see what happens. Obviously if the house is all men or all women, the sex of the last person beamed up will be determined. If there are equal numbers of men and women, then by symmetry, the probability that the last person beamed up is a woman would be 1/2. So the simplest interesting case is, say, one man and two women.

In that case, if the man is beamed up first (probability: 1/3), the last one will be a woman. Suppose a woman is beamed up first; if she is followed by a man (who is then beamed back down), we are down to the symmetric case where the probability of ending with a woman is 1/2. Finally, if a second woman follows the first (probability $\frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$), the man will be last to be beamed up. Putting the cases together, we get probably 1/2 that the last person beamed up is a woman. Is it possible that 1/2 is the answer no matter how many men and women are present, as long as there's at least one of each?

Looking more closely at the above analysis, it seems that the sex of the last person beamed up is determined by the *next-to-last* saucer—the one that reduces the house to one sex. To see why this is so, it is useful to imagine that the Xylofonian acquisition process operates the following way: Each time a flying saucer arrives, the current inhabitants of the house arrange themselves in a uniformly random permutation, from which they are beamed up left to right.

For example, if the inhabitants at one saucer's arrival consist of males m_1 and m_3 and females f_2, f_3 and f_5, and they arrange themselves "f_3, f_5, m_1, f_2, m_3," then the saucer will beam up f_3, f_5, and m_1, then will beam m_1 back down again and take off with just the females f_3 and f_5. The remaining folks, m_1, m_3, and f_2, will now re-permute themselves in anticipation of the next saucer's arrival.

We see that a saucer will be the next to last just when the permutation it encounters consists of all men followed by all women, or all women followed by all men. But no matter how many of each sex are in the house at this point, these two events are equally likely! Why? Because if we simply reverse the order of a such a permutation, we go from all-men-then-all-women to all-women-then-all-men, and vice-versa.

There's just one more observation to make: If both men and women are present initially, then one saucer will never do, thus there always will be a next-to-last saucer. When that comes—even though we do not know in advance which saucer it will be—it is equally likely to depart with the rest of the men, or the rest of the women.

Gasoline Crisis

You need to make a long circular automobile trip during a gasoline crisis. Inquiries have ascertained that the gas stations along the route contain just enough fuel to make it all the way around. If you have an empty tank but can start at a station of your choice, can you complete a clockwise round trip?

Solution:

Yes. The trick is to imagine that you begin at station 1 (say) with *plenty* of fuel, then proceed around the route, emptying each station as you go. When you return to station 1, you will have the same amount of fuel in your tank as when you started.

As you do this, keep track of how much fuel you have left as you pull into each station; suppose that this quantity is minimized at station k. Then, if you start at station k with an empty tank, you will not run out of fuel between stations. ♡

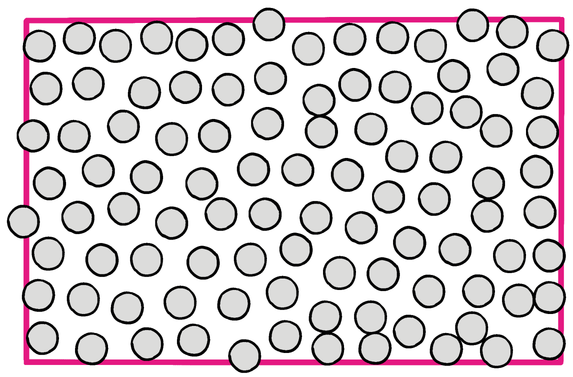
Coins on the Table

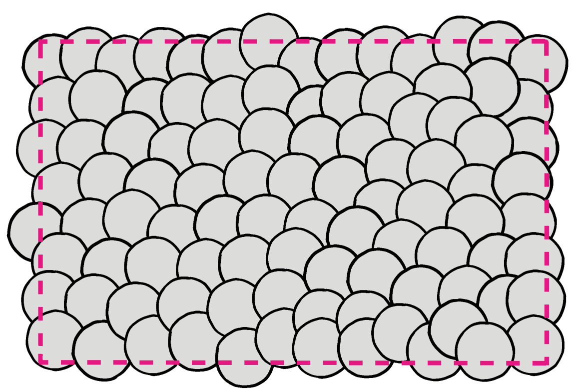
One hundred quarters lie on a rectangular table, in such a way that no more can be added without overlapping. (We allow a quarter to extend over the edge, as long as its center is on the table.)

Prove that you can start all over again and cover the whole table with 400 quarters! (This time we allow overlap *and* overhang).

Solution:

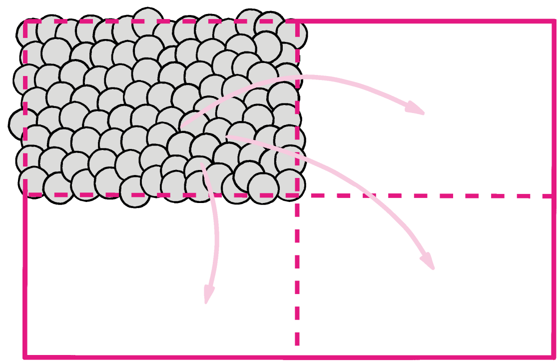
Let us observe first that if we double the radius (say, from $1''$ to $2''$) of each of the original coins, the result will be to cover the whole table. Why? Well, if a point P isn't covered, it must be $2''$ or more from any coin center, thus a (small) coin placed with its center at P would have fit into the original configuration. (See the first two figures below for an example of an original configuration, and what happens when the coins are expanded.)





Now, if we could replace each big coin by four small ones that cover the same area, we'd be done—but we can't.

But rectangles *do* have the property that they can be partitioned into four copies of themselves. So, let us shrink the whole picture (of big coins covering the table) by a factor of two in each dimension, and use four copies (as in the next figure) of the new picture to cover the original table!



Coins in a Row

On a table is a row of 50 coins, of various denominations. Alix picks a coin from one of the ends and puts it in her pocket; then Bert chooses a coin from one of the (remaining) ends, and the alternation continues until Bert pockets the last coin.

Prove that Alix can play so as to guarantee at least as much money as Bert.

Solution:

This puzzle resists the most obvious approaches. It's easy to check that Alix could do quite badly by always choosing the most valuable coin, or the coin that exposes the less valuable coin to Bert, or any combination of these. Basically, if she only looks a move or two ahead, she's in trouble.

In fact, for Alix to play *optimally*, she needs to analyze all the possible situations that may later arise. This can be done by a technique called "dynamic programming."

But we were not asked to provide an optimal strategy for Alix, just a strategy that guarantees her at least half the money. Experimenting with 4 or 6 coins instead of 50 might lead you to the following key observation.

Suppose the coins alternate quarter, penny, quarter, penny, and so forth, ending (since 50 is even) in a penny. Then Alix can get *all* the quarters! In fact, no matter what the coins are, if we number the coins from 1 to 50 left to right, Alix can take all the odd-numbered ones—or all the even-numbered ones.

But wait a minute—one of those two groups of coins must contain at least half the money! ♡

Powers of Roots

What is the first digit after the decimal point in the number $(\sqrt{2}+\sqrt{3})$ to the billionth power?

Solution:

If you try entering $(\sqrt{2}+\sqrt{3})^{1,000,000,000}$ in your computer, you're likely to find that you get only the dozen or so most significant figures; that is, you don't get an accurate enough answer to see what happens after the decimal point.

But you can try smaller powers and see what happens. For example, the decimal expansion of $(\sqrt{2}+\sqrt{3})^{10}$ begins 95049.9999895. A bit of experimentation shows that each even power of $(\sqrt{2}+\sqrt{3})$ seems to be just a hair below some integer. Why? And by how much?

Let's try $(\sqrt{2}+\sqrt{3})^2$, which is about 9.9. If we play with $10-(\sqrt{2}+\sqrt{3})^2$ we discover that it's equal to $(\sqrt{3}-\sqrt{2})^2$. Aha!

Yes, $(\sqrt{3}+\sqrt{2})^{2n}+(\sqrt{3}-\sqrt{2})^{2n}$ is always an integer, because when you expand it, the terms with odd powers cancel and the terms with even powers are integers. But of course $(\sqrt{3}-\sqrt{2})^{2n}$ is very small, about 10^{-n}, so the first roughly n digits of $(\sqrt{3}+\sqrt{2})^{2n}$ after the decimal point are all 9's.

Coconut Classic

Five men and a monkey, marooned on an island, collect a pile of coconuts to be divided equally the next morning. During the night, however, one of the men decides he'd rather take his share now. He tosses one coconut to the monkey and removes exactly 1/5 of the remaining coconuts for himself. A second man does the same thing, then a third, fourth, and fifth.

The following morning the men wake up together, toss one more coconut to the monkey, and divide the rest equally. What's the least original number of coconuts needed to make this whole scenario possible?

Solution:

You can solve this by considering two men instead of five, then three, then guessing. But the following argument is irresistible, once found.

There's an elegant "solution" to the puzzle if you allow negative numbers of coconuts(!). The original pile has -4 coconuts; when the first man tosses the monkey a coconut, the pile is down to -5 but when he "takes" 1/5 of this he is actually adding a coconut, restoring the pile to -4 coconuts. Continuing this way, come morning there are still -4 coconuts; the monkey takes one and the men split up the remaining -5.

It's not obvious that this observation does us any good, but let's consider what happens if there is no monkey; each man just takes 1/5 of the pile he encounters, and in the morning there's a multiple of 5 coconuts left that the men can split. Since each man has reduced the pile by the fraction 4/5, the original number of coconuts must have been a multiple of 5^6 (which shrinks to $4^5 \cdot 5$ by morning).

All we need to do now is add our two pseudo-solutions, by starting with $5^6 - 4 = 15,621$ coconuts. Then the pile reduces successively to $4 \cdot 5^5 - 4$ coconuts, $4^2 \cdot 5^4 - 4$, $4^3 \cdot 5^3 - 4$, $4^4 \cdot 5^2 - 4$, and $4^5 \cdot 5 - 4$. When the monkey gets his morning coconut, we have $4^5 \cdot 5 - 5$ coconuts, a multiple of 5, for the men to split. This is best possible because we needed $5^5 \cdot k - 4$ coconuts to start with, just to have an integer number come morning, and to get $4^5 \cdot k - 5$ to be a multiple of 5 we needed k to be a multiple of 5.

Doubtless, many theorems in mathematics were "discovered" when someone played around with small numbers and then saw a pattern that turned out to be a provable phenomenon.

Here's a theorem that could well have been found that way. Suppose you are running a dojo with an even number n of students. Each day you pair the students up for one-on-one sparring. Can you do this in such a way that over a period of days, each student spars with each other student exactly once?

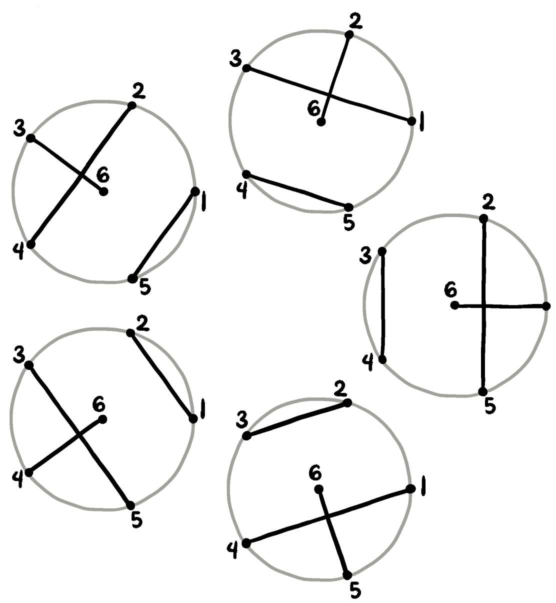
Theorem. . *For any even positive integer* n *there is a set of pairings ("perfect matchings") of the numbers* $\{1, 2, \ldots, n\}$ *such that every pair* $\{i, j\}$ *appears in exactly one pairing.*

A check of small numbers suggests that this seems to work: For $n = 2$, for instance, there is just the one pairing consisting of the pair $\{1, 2\}$, and for $n = 4$ we can (in fact, must) take the pairings $\{\{1, 2\}, \{3, 4\}\}$, $\{\{1, 3\}, \{2, 4\}\}$, and $\{\{1, 4\}, \{2, 3\}\}$.

For $n = 6$, though, we have choices to make. Is there a nice way to make them?

In fact, there are several; my favorite is the following. We know student n has to be paired with every other student; let's put her in the middle of a circle, with the rest of the students spaced equally around the circle. In the ith of our $n-1$ pairings, student n is paired with student i; draw a radius from n to i. The rest of the students are paired by line segments that run perpendicular to that radius (see the figure below).

Since $n-1$ is odd, all the radii from n to other students are at different angles; it follows that no two students are paired twice, and since the eventual number of pairs is $(n-1) \times n/2 = \binom{n}{2}$, every pair is accounted for. ♡



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