The Sine and Cosine Functions

The Graphs of Sine and Cosine

In the last lecture, we defined the quantities $\sin\theta$ and $\cos\theta$ for all angles θ. Today we explore the sine and cosine functions, their properties, their derivatives, and variations on those two functions.

By now, you should have memorized the values of $\sin\theta$ and $\cos\theta$ for all of the special angles. For the purposes of completeness, we recreate the tables of these values from the last lecture below:

θ	$\cos\theta$	$\sin\theta$
0	1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	-$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	-$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
π	-1	0

θ	$\cos\theta$	$\sin\theta$
$\frac{7\pi}{6}$	-$\frac{\sqrt{3}}{2}$	-$\frac{1}{2}$
$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{2}$	0	-1
$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
2π	1	0

Clearly, we can plot the function $f(x) = \sin x$ and the function $g(x) = \cos x$ (replacing θ with x) using the numerical tables above. We note that $\frac{\sqrt{2}}{2} \approx 0.71$, $\frac{\sqrt{3}}{2} \approx 0.87$, $\pi \approx 3.14$, $\frac{\pi}{2} \approx 1.57$, $\frac{\pi}{4} \approx 0.79$, $\frac{\pi}{6} \approx 0.52$, and $\frac{\pi}{3} \approx 1.05$.

The resulting graph of $f(x) = \sin x$ looks like the following: we begin at the origin and with a positive slope of 1. As x increases, the slope of the sine function decreases, so that the derivative is 0 when we get to the point $(\frac{\pi}{2}, 1)$, which is a local maximum for $\sin x$. The function then begins to decrease, first with a gentle slope, but by the point when the graph crosses the x-axis, at the point $(\pi, 0)$, the slope of the graph of the sine function is -1. This point is also an inflection point, so the graph, which up until now was always concave down, now begins to be concave up. The slope of the graph starts becoming less negative, so that at the point $(\frac{3\pi}{2}, -1)$ the graph has zero slope. This point is at a local minimum for the sine function, so the graph now rises again, its slope increasing, until at $(2\pi, 0)$ its slope is 1 again.

Instead of continuing the graph of $f(x) = \sin x$ in both directions, let us now attempt to sketch the graph of $g(x) = \cos x$. We begin at the point $(0, 1)$. Here, the slope of the cosine function will be zero, and the function will have a local maximum, so the graph will begin to drop, first with a gentle negative slope, until finally it reaches the point $(\frac{\pi}{2}, 0)$, where its slope is -1. This point is an inflection point, so now the graph goes from concave down to concave up. The slope of the graph increases, so that at $(\pi, -1)$ we have a critical point of the function, and a local minimum. The graph now starts to rise, and by the point it reaches the x-axis, which is $(\frac{3\pi}{2}, 0)$, the slope of the graph is 1, and we have another inflection point, this time from concave up to concave down. The graph continues to rise, but more and more slowly, and then, at $(2\pi, 1)$, the slope is zero and we are at a local maximum again.

What about the graphs for x greater than 2π and for negative x? Here we use the fact that angles whose arcs have the same endpoint have the same sine and cosine. This means that, every time x changes by 2π (the number of radians in a circle), the graphs of sine and cosine repeat themselves. So we sketch in the graphs of $\sin x$ and $\cos x$ for x greater than 2π and negative x by simply continuing the graph just as it had started and stopped. The only new point to make here is that $(0, 0)$ and $(0, 2\pi)$ are now clearly inflection points, where the graphs goes from concave up to concave down.

The Properties of Sine and Cosine

Let us now list in algebraic form the properties of the sine and cosine functions:

- **Periodicity:** Both $\sin x$ and $\cos x$ have the property that they repeat themselves every time x increases (or decreases) by 2π. Algebraically, we write this property as

$$\sin(x + 2\pi) = \sin x \qquad \text{and} \qquad \cos(x + 2\pi) = \cos x.$$

The length of this repeating block of values is called the period of the function. So the period for both functions is 2π.

- **Boundedness:** Unlike polynomial functions (with the exception of constant functions), the function $\sin x$ and $\cos x$ have both a maximum value and a minimum value. For both functions, the maximum value of the function is 1, and the minimum value is -1. Both functions take all values between -1 and 1, so the range of both functions is $-1 \leq y \leq 1$. A function which has both a maximum value and a minimum value (not just a local maximum and a local minimum) is called a bounded function. In the case of $\sin x$ and $\cos x$, since they are both bounded and periodic, we can talk about their amplitude, the largest value that $|\sin x|$ and $|\cos x|$ can take, or, equivalently, the largest vertical distance the points on the graphs of these two functions can get from the x-axis. The amplitude of both functions is 1.

- **Shifts:** You may have noticed from the numerical tables of $\sin x$ and $\cos x$, or from their graphs, that the values of $\sin x$ seem to trail the values of $\cos x$. Specifically, the graph of $\cos x$ is the graph of $\sin x$ shifted backwards a distance of $\frac{\pi}{2}$. We write this property algebraically as

$$\cos x = \sin\left(x + \frac{\pi}{2}\right).$$

  So the functions $\sin x$ and $\cos x$ are very closely related to each other.

- **Evenness and Oddness:** Looking at the graph of $\sin x$, we see that it has point symmetry at the origin, and, specifically, that it passes through the origin. This means that $\sin x$ is an odd function, which we write algebraically as:

$$\sin(-x) = -\sin x.$$

  You can test this fact using the numerical tables above. As for $\cos x$, we see that its graph has an axis of symmetry along the y-axis, so it is an even function, which we write algebraically as:

$$\cos(-x) = \cos x.$$

  You can also check this property using the tables above.

The Derivatives of Sine and Cosine

Since this is a calculus class, we now have the opportunity to study some more interesting properties of the sine and cosine functions, specifically their derivatives. We are going to sketch the graph of the sine function by hand, using the techniques of graphing derivatives that we learned earlier in the class.

First, it is worth noting that, since $\sin x$ and $\cos x$ are periodic functions, their first derivatives, and all higher derivatives, are also periodic, of the same period. This is because as the values of the function repeat, so do the slopes of the tangent lines to the graph. So, when sketching the derivative of $\sin x$ or $\cos x$, we need only sketch one interval of 2π of the derivative. We choose to do this for the interval $[0, 2\pi]$.

Now sketch the function $\sin x$ from $x = 0$ to $x = 2\pi$. To sketch the derivative, we first need to find the critical points. When we first sketched $\sin x$, we noted that the slope of the graph is zero at $x = \frac{\pi}{2}$ and at $x = \frac{3\pi}{2}$. So we know that the graph of the derivative of $\sin x$ touches the x-axis at these two x-values. Now we need to find the inflection points. We noted that we have three inflection points in this closed interval: at $x = 0$, $x = \pi$, and at $x = 2\pi$. We also noted that the value of the derivative of $\sin x$ is 1 at $x = 0$, -1 at $x = \pi$, and 1 again at $x = 2\pi$. This tells us that the graph of the derivative has a local maximum at the point $(1, 0)$, a local minimum at the point $(\pi, -1)$, and a local maximum again at $(2\pi, 1)$. We plot these points, and then sketch the graph of the derivative of $\sin x$ using all of this information. The curve that we get looks very familiar: it is the graph of $\cos x$. The derivative of $\sin x$ is $\cos x$:

$$\frac{\mathrm{d}}{\mathrm{d}x} \sin x = \cos x.$$

So, apparently, the functions $\sin x$ and $\cos x$ are related in yet another way!

Now, let us sketch the derivative of $\cos x$. First, plot the graph of $\cos x$ over the closed interval $[0, 2\pi]$. Again, we need to find the critical points of $\cos x$. There are three critical points on this interval: at $x = 0$, $x = \pi$, and $x = 2\pi$. So the graph of the derivative of $\cos x$ touches the x-axis on this interval at three points: $(0, 0)$, $(\pi, 0)$, and $(2\pi, 0)$. Now we look for the inflection points, and we two of them on this interval: one at $x = \frac{\pi}{2}$, where the slope of the graph is apparently -1, and one at $x = \frac{3\pi}{2}$, where the derivative is 1. So the graph of the derivative will pass through the point $(\frac{\pi}{2}, -1)$, where it has a local minimum, and the point $(\frac{3\pi}{2}, 1)$, where it has a local maximum. Now we sketch the graph of the derivative of $\cos x$ using all of the information above, and we get a curve which looks very familiar, but not quite like any curve we have seen before. It appears to be the mirror image of the graph of $\sin x$ in the x-axis. What does this mean algebraically? It means that this is the graph of the function $-\sin x$, the function that you get by reversing the sign of all of the values of $\sin x$. So, the derivative of $\cos x$ is $-\sin x$:

$$\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -\sin x.$$

Another relationship between $\sin x$ and $\cos x$ is revealed.

Knowing the first derivatives of $\sin x$ and $\cos x$, we can now find their higher derivatives. The second derivative of $\sin x$ is the first derivative of $\cos x$, which is $-\sin x$. To get the third derivative, we apply the constant multiple rule:

$$\frac{\mathrm{d}^3}{\mathrm{d}x^3} \sin x = \frac{\mathrm{d}}{\mathrm{d}x}(-\sin x) = -\frac{\mathrm{d}}{\mathrm{d}x} \sin x = -\cos x.$$

So the third derivative of $\sin x$ is $-\cos x$. The fourth derivative of $\sin x$ also comes from an application of the constant multiple rule:

$$\frac{\mathrm{d}^4}{\mathrm{d}x^4} \sin x = \frac{\mathrm{d}}{\mathrm{d}x}(-\cos x) = -\frac{\mathrm{d}}{\mathrm{d}x} \cos x = -(-\sin x) = \sin x.$$

So the fourth derivative of $\sin x$ is itself. That means that its fifth derivative is $\cos x$, its sixth derivative is $-\sin x$, and so on: the higher derivatives of $\sin x$ are period in yet another way. The higher derivatives of $\cos x$ also show this periodicity, which is illustrated below:

$f(x)$	$f'(x)$	$f''(x)$	$f^{(3)}(x)$	$f^{(4)}(x)$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	$\cos x$

Generalized Sine and Cosine Functions

Finally, we want to discuss more general sine and cosine functions. Specifically, a generalized sine function is a function of the form

$$f(x) = A \sin(kx),$$

and a generalized cosine function is a function of the form

$$g(x) = A \cos(kx).$$

Generalized sine and cosine functions are both periodic and bounded, that is, they have an amplitude. In the cases above, the amplitudes of these generalized sine and cosine functions are given by $|A|$. The periods of these functions are given by $\frac{2\pi}{k}$. Compare these formulae for amplitude and period to the amplitudes and periods of the original sine and cosine functions, where $A = 1$ and $k = 1$. Graphically, increasing A has the effect of stretching the graphs of sine and cosine vertically, and increasing k has the effect of shrinking the graphs of sine and cosine horizontally. We will discuss this stretching and shrinking in a more general context in a later class.

The derivative of the generalized sine function above is

$$\frac{\mathrm{d}}{\mathrm{d}x}(A \sin(kx)) = Ak \cos(kx),$$

and the derivative of the generalized cosine function above is

$$\frac{\mathrm{d}}{\mathrm{d}x}(A\cos(kx)) = -Ak\sin(kx).$$

So, the formulae for the derivatives is similar to the original case, except that we pull a k out of the sine and cosine functions in the process of taking the derivative.

As an example, take $f(x) = -3\sin(4x)$. The amplitude of this generalized sine function, its maximum vertical distance from the x-axis is $|-3| = 3$. Its period is

$$\frac{2\pi}{k} = \frac{2\pi}{4} = \frac{\pi}{2},$$

so the graph of this sine function is much taller vertically and much narrower horizontally than the graph of the original sine function. Finally, its derivative is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = -3 \cdot 4\cos(4x) = -12\cos(4x).$$

If you were to plot the graph of $f(x) = -3\sin(4x)$, and then sketch its derivative, do you think you would get the graph of the function above? Try it and find out.