The First and Second Derivatives

The Meaning of the First Derivative

At the end of the last lecture, we knew how to differentiate any polynomial function. Polynomial functions are the first functions we studied for which we did not talk about the shape of their graphs in detail. To rectify this situation, in today's lecture, we are going to formally discuss the information that the first and second derivatives give us about the shape of the graph of a function.

The first derivative of the function $f(x)$, which we write as $f'(x)$ or as $\frac{\mathrm{d}f}{\mathrm{d}x}$, is the slope of the tangent line to the function at the point x. To put this in non-graphical terms, the first derivative tells us how whether a function is increasing or decreasing, and by how much it is increasing or decreasing. This information is reflected in the graph of a function by the slope of the tangent line to a point on the graph, which is sometimes describe as the slope of the function. Positive slope tells us that, as x increases, $f(x)$ also increases. Negative slope tells us that, as x increases, $f(x)$ decreases. Zero slope does not tell us anything in particular: the function may be increasing, decreasing, or at a local maximum or a local minimum at that point. Writing this information in terms of derivatives, we see that:

- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) > 0$, then $f(x)$ is an increasing function at $x = p$.
- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) < 0$, then $f(x)$ is a decreasing function at $x = p$.
- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 0$, then $x = p$ is called a critical point of $f(x)$, and we do not know anything new about the behavior of $f(x)$ at $x = p$.

For example, take $f(x) = 3x^3 - 6x^2 + 2x - 1$. The derivative of $f(x)$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x} = 9x^2 - 12x + 2.$$

At $x = 0$, the derivative of $f(x)$ is therefore 2, so we know that $f(x)$ is an increasing function at $x = 0$. At $x = 1$, the derivative of $f(x)$ is

$$\frac{\mathrm{d}f}{\mathrm{d}x}(1) = 9 \cdot 1^2 - 12 \cdot 1 + 2 = 9 - 12 + 2 = -1,$$

so $f(x)$ is a decreasing function at $x = 1$.

The Meaning of the Second Derivative

The second derivative of a function is the derivative of the derivative of that function. We write it as $f''(x)$ or as $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$. While the first derivative can tell us if the function is increasing or decreasing, the second derivative tells us if the first derivative is increasing or decreasing. If the second derivative is positive, then the first derivative is increasing, so that the slope of the tangent line to the function is increasing as x increases. We see this phenomenon graphically as the curve of the graph being concave up, that is, shaped like a parabola open upward. Likewise, if the second derivative is negative, then the first derivative is decreasing, so that the slope of the tangent line to the function is decreasing as x increases. Graphically, we see this as the curve of the graph being concave down, that is, shaped like a parabola open downward. At the points where the second derivative is zero, we do not learn anything about the shape of the graph: it may be concave up or concave down, or it may be changing from concave up to concave down or changing from concave down to concave up. So, to summarize:

- if $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) > 0$ at $x = p$, then $f(x)$ is concave up at $x = p$.
- if $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) < 0$ at $x = p$, then $f(x)$ is concave down at $x = p$.
- if $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) = 0$ at $x = p$, then we do not know anything new about the behavior of $f(x)$ at $x = p$.

For an example of finding and using the second derivative of a function, take $f(x) = 3x^3 - 6x^2 + 2x - 1$ as above. Then $f'(x) = 9x^2 - 12x + 2$, and $f''(x) = 18x - 12$. So at $x = 0$, the second derivative of $f(x)$ is -12, so we know that the graph of $f(x)$ is concave down at $x = 0$. Likewise, at $x = 1$, the second derivative of $f(x)$ is

$$f''(1) = 18 \cdot 1 - 12 = 18 - 12 = 6,$$

so the graph of $f(x)$ is concave up at $x = 1$.

Critical Points and the Second Derivative Test

We learned before that, when x is a critical point of the function $f(x)$, we do not learn anything new about the function at that point: it could increasing, decreasing, a local maximum, or a local minimum. We can often use the second derivative of the function, however, to find out when x is a local maximum or a local minimum.

Recall that x is a critical point of a function when the slope of the function is zero at that point. Now, suppose that x is a critical point and the second derivative of the function at that point is positive. The positive second derivative at x tells us that the derivative of $f(x)$ is increasing at that point and, graphically, that the curve of the graph is concave up at that point. The only way to sketch the graph of a function at a point where the slope of the function is zero but the graph is concave up is to make that point a local minimum of the function. So, if x is a critical point of $f(x)$ and the second derivative of $f(x)$ is positive, then x is a local minimum of $f(x)$.

Likewise, if x is a critical point of $f(x)$ and the second derivative of $f(x)$ is negative, then the slope of the graph of the function is zero at that point, but the curve of the graph is concave down. The only way to draw a graph like this to make the point x a local maximum of the function. Hence we get that if x is a critical point of $f(x)$ and the second derivative of $f(x)$ is negative, then x is a local maximum of $f(x)$.

When x is a critical point of $f(x)$ and the second derivative of $f(x)$ is zero, then we learn no new information about the point. The point x may be a local maximum or a local minimum, and the function may also be increasing or decreasing at that point.

The three cases above, when the second derivative is positive, negative, or zero, are collectively called the second derivative test for critical points. The second derivative test gives us a way to classify critical point and, in particular, to find local maxima and local minima. To summarize the second derivative test:

- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 0$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) > 0$, then $f(x)$ has a local minimum at $x = p$.
- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 0$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) < 0$, then $f(x)$ has a local maximum at $x = p$.
- if $\frac{\mathrm{d}f}{\mathrm{d}x}(p) = 0$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(p) = 0$, then we learn no new information about the behavior of $f(x)$ at $x = p$.

For example, take $g(x) = x^3 - 9x^2 + 15x - 7$, and let us find the critical points of $g(x)$ and if any of its critical points are local maxima or local minima. The derivative of $g(x)$ is

$$g'(x) = 3x^2 - 18x + 15.$$

The critical points of $g(x)$ are precisely the values of x where the derivative of $g(x)$ is 0, so we set the formula above equal to 0 and solve the resulting quadratic equation:

$$\begin{aligned} 3x^2 - 18x + 15 &= 0 \\ x^2 - 6x + 5 &= 0 \\ (x-1)(x-5) &= 0 \\ x = 1 \quad \text{or} \quad x &= 5. \end{aligned}$$

So the critical points of $g(x)$ are $x = 1$ and $x = 5$. We now want to apply the second derivative test, and to do that we need to find a formula for the second derivative:

$$g''(x) = 6x - 18.$$

So the second derivative of $g(x)$ at $x = 1$ is

$$g''(1) = 6 \cdot 1 - 18 = 6 - 18 = -12,$$

and the second derivative of $g(x)$ at $x = 5$ is

$$g''(5) = 6 \cdot 5 - 18 = 30 - 18 = 12.$$

Therefore the second derivative test tells us that $g(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 5$.

Inflection Points

Finally, we want to discuss inflection points in the context of the second derivative. We recall that the graph of a function $f(x)$ has an inflection point at x if the graph of the function goes from concave up to concave down at that point, or if the graph of the function goes from concave down to concave up at that point. Clearly then, an inflection point can only happen where at points where the second derivative is 0, because otherwise the point would the graph would be either completely concave up or completely concave down at that point. Just like in the case of local maxima and local minima and the first derivative, however, the presence of a point where the second derivative of a function is 0 does not automatically tell us that the point is an inflection point. For example, take $f(x) = x^4$. Then $f'(x) = 4x^3$ and $f''(x) - 12x^2$, so $f''(0) = 0$, but if we were sketch the function $f(x) = x^4$, it becomes clear that $x = 0$ is not an inflection point for $f(x)$, since $f(x)$ has the familiar U-shape of even positive power functions. So, while the second derivative can tell us a lot about the shape of the graph of a function, it cannot tell us everything: it cannot tell us if the graph of a function has an inflection point; it can only tell us where it might have an inflection point. Can you think of a test like the second derivative test that we could use to conclusively find inflection points?