Difference Equations
to
Differential Equations

Section 3.5

Differentiation of Trigonometric Functions

We now take up the question of differentiating the trigonometric functions. We will start with the sine function. From Section 3.2, we know that

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}. \tag{3.5.1}$$

From the addition formula for sine we have

$$\sin(x+h) = \sin(x)\cos(h) + \sin(h)\cos(x), \tag{3.5.2}$$

and so (3.5.1) becomes

$$\frac{d}{dx}\sin(x) = \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}. \tag{3.5.3}$$

Now

$$\begin{aligned} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h} &= \frac{\sin(x)(\cos(h) - 1) + \cos(x)\sin(h)}{h} \\ &= \sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right). \end{aligned}$$

Thus

$$\frac{d}{dx}\sin(x) = \sin(x)\lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x)\lim_{h \to 0} \frac{\sin(h)}{h}. \tag{3.5.4}$$

Our problem then comes down to evaluating the two limits in (3.5.4). The second of these turns out to be the key, so we will begin with it.

For $0 < h < \frac{\pi}{2}$, consider the point $C = (\cos(h), \sin(h))$ on the unit circle centered at the origin. We first repeat an argument from Section 2.4 to show that $\sin(h) < h$: If we let $A = (0,0)$ and $B = (1,0)$, as in Figure 3.5.1, then the area of $\triangle ABC$ is

$$\frac{1}{2}\sin(h).$$

The area of the sector of the circle cut off by the arc from B to C is the fraction $\frac{h}{2\pi}$ of the area of the entire circle; hence, this area is

$$\frac{h}{2\pi}\pi = \frac{h}{2}.$$

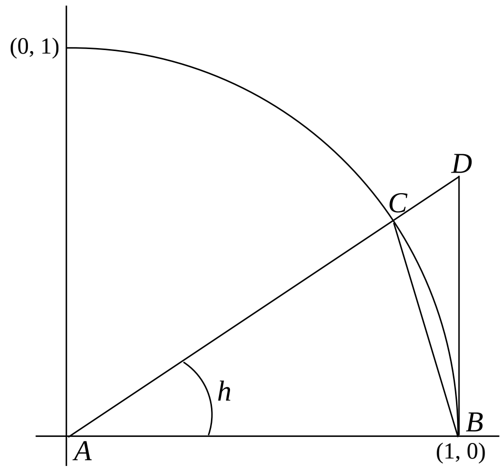

Figure 3.5.1

Since $\triangle ABC$ is contained in this section, we have

$$\frac{1}{2}\sin(h) < \frac{h}{2}, \tag{3.5.5}$$

or simply

$$\sin(h) < h. \tag{3.5.6}$$

Now let $D = (1, \tan(h))$, the point where the line passing through A and C intersects the line perpendicular to the x-axis passing through B. Then $\triangle ABD$ has area

$$\frac{1}{2}\tan(h) = \frac{\sin(h)}{2\cos(h)}.$$

Since $\triangle ABD$ contains the sector of the circle considered above, we have

$$\frac{h}{2} < \frac{\sin(h)}{2\cos(h)}, \tag{3.5.7}$$

or

$$h < \frac{\sin(h)}{\cos(h)}. \tag{3.5.8}$$

Putting inequalities (3.5.6) and (3.5.8) together gives us

$$\sin(h) < h < \frac{\sin(h)}{\cos(h)}. \tag{3.5.9}$$

Dividing through by $\sin(h)$ yields

$$1 < \frac{h}{\sin(h)} < \frac{1}{\cos(h)}, \tag{3.5.10}$$

which, after taking reciprocals, gives us

$$1 > \frac{\sin(h)}{h} > \cos(h). \tag{3.5.11}$$

Now, finally, we can see where all of this has been heading. Since

$$\lim_{h \to 0^+} \cos(h) = 1,$$

(3.5.11) implies that we must have

$$\lim_{h \to 0^+} \frac{\sin(h)}{h} = 1. \tag{3.5.12}$$

To check the limit from the other side, we make use of the identity $\sin(-x) = -\sin(x)$. Letting $t = -h$, we have

$$\lim_{h \to 0^-} \frac{\sin(h)}{h} = \lim_{h \to 0^-} \frac{-\sin(h)}{-h} = \lim_{h \to 0^-} \frac{\sin(-h)}{-h} = \lim_{t \to 0^+} \frac{\sin(t)}{t} = 1. \tag{3.5.13}$$

Together (3.5.12) and (3.5.13) give us the following proposition.

Proposition

$$\lim_{h \to 0} \frac{\sin(h)}{h} = 1. \tag{3.5.14}$$

With this result, we may now compute

$$\begin{aligned}
\lim_{h \to 0} \frac{1 - \cos(h)}{h} &= \lim_{h \to 0} \left(\frac{1 - \cos(h)}{h}\right)\left(\frac{1 + \cos(h)}{1 + \cos(h)}\right) \\
&= \lim_{h \to 0} \frac{1 - \cos^2(h)}{h(1 + \cos(h))} \\
&= \lim_{h \to 0} \frac{\sin^2(h)}{h(1 + \cos(h))} \\
&= \lim_{h \to 0} \left(\frac{\sin(h)}{h}\right)\left(\frac{\sin(h)}{1 + \cos(h)}\right) \\
&= \lim_{h \to 0} \frac{\sin(h)}{h} \lim_{h \to 0} \frac{\sin(h)}{1 + \cos(h)} \\
&= (1)\left(\frac{0}{2}\right) = 0.
\end{aligned}$$

Proposition

$$\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0. \tag{3.5.15}$$

Of course, from (3.5.15) we have

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} = -\lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0. \tag{3.5.16}$$

Putting (3.5.14) and (3.5.16) into (3.5.4) gives us

$$\begin{aligned} \frac{d}{dx} \sin(x) &= \sin(x) \lim_{h \to 0} \frac{\cos(h) - 1}{h} + \cos(x) \lim_{h \to 0} \frac{\sin(h)}{h} \\ &= \sin(x)(0) + \cos(x)(1) \\ &= \cos(x). \end{aligned}$$

Proposition The function $f(x) = \sin(x)$ is differentiable for all x in $(-\infty, \infty)$ with

$$\frac{d}{dx} \sin(x) = \cos(x). \tag{3.5.17}$$

The derivatives of the other trigonometric functions now follow with the help of some basic identities. Since $\cos(x) = \sin(x + \frac{\pi}{2})$ and $\cos(x + \frac{\pi}{2}) = -\sin(x)$, it follows that

$$\frac{d}{dx} \cos(x) = \frac{d}{dx} \sin\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) \frac{d}{dx}\left(x + \frac{\pi}{2}\right) = \cos\left(x + \frac{\pi}{2}\right) = -\sin(x).$$

The other four derivatives are as follows:

$$\begin{aligned} \frac{d}{dx} \tan(x) &= \frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right) \\ &= \frac{\cos(x) \frac{d}{dx} \sin(x) - \sin(x) \frac{d}{dx} \cos(x)}{\cos^2(x)} \\ &= \frac{\cos(x)\cos(x) - \sin(x)(-\sin(x))}{\cos^2(x)} \\ &= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\ &= \frac{1}{\cos^2(x)} \\ &= \sec^2(x), \end{aligned}$$

$$\begin{aligned} \frac{d}{dx} \cot(x) &= \frac{d}{dx}\left(\frac{\cos(x)}{\sin(x)}\right) \\ &= \frac{\sin(x) \frac{d}{dx} \cos(x) - \cos(x) \frac{d}{dx} \sin(x)}{\sin^2(x)} \end{aligned}$$

$$
\begin{aligned}
&= \frac{\sin(x)(-\sin(x)) - \cos(x)\cos(x)}{\sin^2(x)} \\
&= \frac{-(\sin^2(x) + \cos^2(x))}{\sin^2(x)} \\
&= -\frac{1}{\sin^2(x)} \\
&= -\csc^2(x),
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dx}\sec(x) &= \frac{d}{dx}(\cos(x))^{-1} \\
&= -(\cos(x))^{-2}\frac{d}{dx}\cos(x) \\
&= \frac{\sin(x)}{\cos^2(x)} \\
&= \left(\frac{1}{\cos(x)}\right)\left(\frac{\sin(x)}{\cos(x)}\right) \\
&= \sec(x)\tan(x),
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d}{dx}\csc(x) &= \frac{d}{dx}(\sin(x))^{-1} \\
&= -(\sin(x))^{-2}\frac{d}{dx}\sin(x) \\
&= -\frac{\cos(x)}{\sin^2(x)} \\
&= -\left(\frac{1}{\sin(x)}\right)\left(\frac{\cos(x)}{\sin(x)}\right) \\
&= \csc(x)\cot(x).
\end{aligned}
$$

The next proposition summarizes these results.

Proposition The derivatives of the trigonometric functions are as follows:

$$\frac{d}{dx}\sin(x) = \cos(x) \tag{3.5.18}$$

$$\frac{d}{dx}\cos(x) = -\sin(x) \tag{3.5.19}$$

$$\frac{d}{dx}\tan(x) = \sec^2(x) \tag{3.5.20}$$

$$\frac{d}{dx}\cot(x) = -\csc^2(x) \tag{3.5.21}$$

$$\frac{d}{dx}\sec(x) = \sec(x)\tan(x) \tag{3.5.22}$$

$$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x) \tag{3.5.23}$$

Example Using the chain rule, we have

$$\frac{d}{dx}\sin(2x) = \cos(2x)\frac{d}{dx}(2x) = 2\cos(2x).$$

Example Using the product rule followed by the chain rule, we have

$$\begin{aligned}\frac{d}{dx}(3\sin(5x)\cos(4x)) &= 3\sin(5x)\frac{d}{dx}\cos(4x) + 3\cos(4x)\frac{d}{dx}\sin(5x)\\ &= 3\sin(5x)(-\sin(4x)\frac{d}{dx}(4x)) + 3\cos(4x)\cos(5x)\frac{d}{dx}(5x)\\ &= -12\sin(5x)\sin(4x) + 15\cos(4x)\cos(5x).\end{aligned}$$

Example Using the chain rule twice, we have

$$\begin{aligned}\frac{d}{dx}\sin^2(3x) &= 2\sin(3x)\frac{d}{dx}\sin(3x)\\ &= 2\sin(3x)\cos(3x)\frac{d}{dx}(3x)\\ &= 6\sin(3x)\cos(3x).\end{aligned}$$

Example Using the product rule followed by the chain rule, we have

$$\begin{aligned}\frac{d}{dt}(t^2\tan(2t)) &= t^2\frac{d}{dt}\tan(2t) + \tan(2t)\frac{d}{dt}t^2\\ &= t^2\sec^2(2t)\frac{d}{dt}(2t) + 2t\tan(2t)\\ &= 2t^2\sec^2(2t) + 2t\tan(2t).\end{aligned}$$

Example Using the chain rule twice, we have

$$\begin{aligned}\frac{d}{dz}\sec^3(3z) &= 3\sec^2(3z)\frac{d}{dz}\sec(3z)\\ &= 3\sec^2(3z)\sec(3z)\tan(3z)\frac{d}{dz}(3z)\\ &= 9\sec^3(3z)\tan(3z).\end{aligned}$$

Example If $f(x) = 8\cot^4(3x^2)$, then

$$\begin{aligned}f(x) &= 32\cot^3(3x^2)\frac{d}{dx}\cot(3x^2)\\ &= 32\cot^3(3x^2)(-\csc^2(3x^2)\frac{d}{dx}(3x^2))\\ &= -192x\cot^3(3x^2)\csc^2(3x^2).\end{aligned}$$

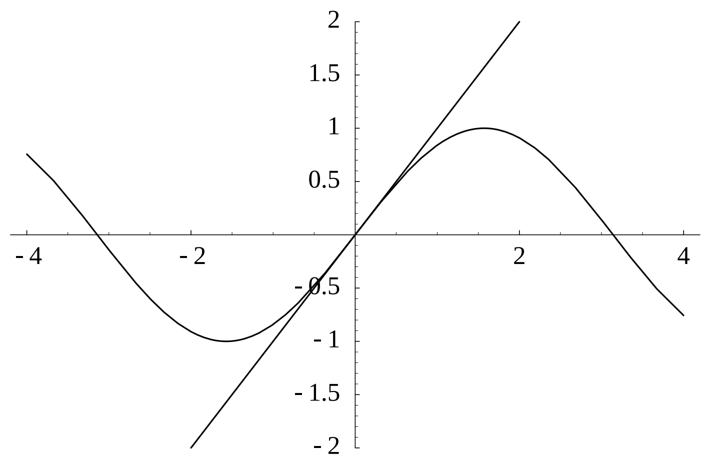

Figure 3.5.2 Graphs of $y = \sin(x)$ and $y = x$

Example If $f(x) = \sin(x)$, then $f(0) = \sin(0) = 0$ and $f'(0) = \cos(0) = 1$. Hence the best affine approximation to $f(x) = \sin(x)$ at $x = 0$ is

$$T(x) = x.$$

This says that for small values of x, $\sin(x) \approx x$ This fact is very useful in many applications where an equation cannot be solved exactly because of the presence of a sine term, but can be solved exactly once the approximation $\sin(x) \approx x$ is made. For example, the formula mentioned in Section 2.2 for the motion of a pendulum undergoing small oscillations was derived after making this approximation. Without this approximation the underlying equation cannot be solved exactly. See Figure 3.5.2 for the graphs of $y = \sin(x)$ and $y = x$.

Final comments on rules of differentiation

With the work of the last three sections we can now routinely differentiate any algebraic function or any combination of an algebraic function with a trigonometric function. In fact, the rules of these last three sections provide algorithms for differentiation which may be incorporated into computer programs. Programs that are capable of performing differentiation in this manner, as well as other types of algebraic procedures, are called symbolic manipulation programs or computer algebra systems. These programs are very useful when working with procedures that require exact knowledge of the formula for the derivative of a given function.

Contrasted to symbolic differentiation is numerical differentiation. Numerical differentiation is performed when we approximate the derivative of a function at a specific point. That is, whereas symbolic differentiation finds a formula for the derivative of a function, which may then be evaluated at any point in its domain to find specific values, numerical differentiation finds a single number which is used as an approximation to the value of the derivative at one given point. For example, if we wish to approximate the derivative of a function f at a point c, we might pick a small value of h, positive or negative, and compute

$$f'(c) \approx \frac{f(c+h) - f(c)}{h}. \tag{3.5.24}$$

Of course, we need some procedure for deciding when h is small enough for (3.5.24) to give an accurate estimate for $f'(c)$. One technique is to use (3.5.24) repeatedly, cutting h in half each time, until the result does not change through the desired number of decimal places. This method is subject to serious roundoff errors due to the loss of significant digits in the numerator when two nearly equal numbers are subtracted (see Problem 13). Hence the numerical approximation of derivatives is not recommended unless it cannot be avoided. Problem 10 suggests an alternative to (3.5.24) which is both more stable for computations and more accurate for a given value of h.

Problems

1. Find the derivative of each of the following functions.

   (a) $f(x) = x^2 \sin(x)$ (b) $g(x) = \cos(4x)$

   (c) $g(t) = 3t\cos(2t)$ (d) $h(s) = \sin^2(s)\cos(s)$

   (e) $f(t) = \sin(3t)\cos(4t)$ (f) $g(z) = \sin^3(4z)$

2. Find the derivative of the dependent variable with respect to the independent variable for each of the following.

   (a) $y = \dfrac{\sin(2x)}{x}$ (b) $x = 3\tan(2t)$

   (c) $x = \sin(4t^2 + 1)$ (d) $y = 4\theta\tan(\theta^2 - 1)$

   (e) $z = \dfrac{1}{\cos(2t)}$ (f) $q = \sec^3(3t)$

   (g) $y = x^2\csc(2x)$ (h) $s = 3t\cot(2t)$

3. Evaluate each of the following.

   (a) $\dfrac{d}{dx}(\sin^2(2x)\cos^2(3x))$ (b) $\dfrac{d}{dx}(\sec(x)\tan(x))$

   (c) $\dfrac{d}{dq}\sec^3(q^2)$ (d) $\dfrac{d}{dt}\left(\dfrac{\sin^2(t)}{\cos(t)}\right)$

   (e) $\dfrac{d}{dz}\sqrt{1 + \sin^2(z)}$ (f) $\dfrac{d}{dr}(r^2\cos(3r^2))$

4. Find the best affine approximation to $f(x) = \tan(2x)$ at 0.

5. Find the best affine approximation to $g(t) = \cos(t)$ at 0.

6. Find the best affine approximation to $f(t) = \sin^2(t)$ at 0.

7. (a) Find the best affine approximation S to $f(x) = \sqrt{1 + x}$ at 0.

   (b) Find the best affine approximation T to $g(x) = \sin(4x)$ at 0.

   (c) Find the best affine approximation U to $h(x) = \sqrt{1 + \sin(4x)}$ at 0.

   (d) What is the relationship between f, g, and h? Is their a similar relationship between S, T, and U?

8. Evaluate the following limits.

   (a) $\lim_{x \to 0} \frac{\sin(2x)}{x}$

   (b) $\lim_{x \to 0} \frac{\sin(2x)}{\sin(3x)}$

   (c) $\lim_{x \to 0} \frac{\tan(x)}{x}$

   (d) $\lim_{h \to 0} \frac{\tan(2h)}{\sin(3h)}$

   (e) $\lim_{x \to 0} \frac{\sin^2(x)}{x}$

   (f) $\lim_{t \to 0} \frac{1 - \cos(t)}{t^2}$

   (g) $\lim_{t \to 0} \frac{\sin^2(3t)}{t^2}$

   (h) $\lim_{\theta \to 0} \frac{\tan^2(5\theta)}{\sin^2(3\theta)}$

9. For each of the following, decide whether or not the given function is $o(h)$ and whether or not it is $O(h)$.

   (a) $f(x) = \sin(x)$

   (b) $f(x) = \sin^2(x)$

   (c) $g(t) = \tan(t)$

   (d) $h(t) = \tan^2(t)$

   (e) $f(t) = 1 - \cos(t)$

   (f) $g(t) = 1 - \cos^2(t)$

10. Given a function f which is differentiable at the point c, define

$$D(h) = \frac{f(c+h) - f(c)}{h}.$$

Then, for small values of h, $f'(c) \approx D(h)$.

   (a) Let $h > 0$. A better approximation for $f'(c)$ than $D(h)$ is given by averaging $D(h)$ and $D(-h)$. Show that if we define

$$D_1(h) = \frac{D(h) + D(-h)}{2},$$

   then

$$D_1(h) = \frac{f(c+h) - f(c-h)}{2h}.$$

   What is $D_1(h)$ geometrically?

   (b) Let $h > 0$. Another approximation that is sometimes used for $f'(c)$ is

$$D_2(h) = \frac{4}{3} D_1\left(\frac{h}{2}\right) - \frac{1}{3} D_1(h).$$

   Show that

$$D_2(h) = \frac{f(c-h) - 8f\left(c - \frac{h}{2}\right) + 8f\left(c + \frac{h}{2}\right) - f(c+h)}{6h}.$$

11. Using $h = 0.00001$, approximate the derivatives of the following functions using $D(h)$, $D_1(h)$, and $D_2(h)$ (from Problem 10) at the indicated points. Compare your answers with the exact values.
    (a) $f(x) = x^2$ at $x = 2$
    (b) $f(x) = \dfrac{1}{x}$ at $x = 2$
    (c) $f(x) = \sin(x)$ at $x = 0$
    (d) $f(x) = 3\sin(x^2)\cos(4x)$ at $x = 0$

12. Compute $D(h)$, $D_1(h)$, and $D_2(h)$ (from Problem 10) for the function $f(x) = |x|$ at $x = 0$. Use $h = 0.001$. Are your answers reasonable? Can you explain them?

13. For $f(x) = x^2$ and $c = 2$, compute the values of

$$e_n = |4 - D(10^{-n})|$$

(see Problem 10) for $n = 1, 2, \ldots, 15$. Note that you are computing the absolute value of the error in approximating $f'(c)$ by $D(h)$ for different values of h. Plot the ordered pairs (n, e_n). Does the absolute value of the error decrease as h decreases? Can you explain your results?