Difference Equations to Differential Equations

Section 4.7

More on Area

In Section 4.1 we motivated the definition of the definite integral with the idea of finding the area of a region in the plane. However, to solve the problem we restricted to a very special type of region, namely, a region lying between the graph of a function f and an interval on the x-axis. We will now consider the more general problem of the area of a region lying between the graphs of two functions.

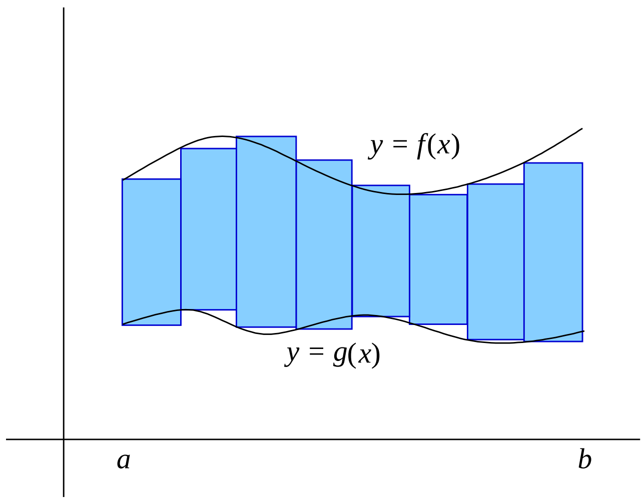

Figure 4.7.1 Approximating the area between $y = f(x)$ and $y = g(x)$

Suppose f and g are functions defined on an interval $[a, b]$ with $g(x) \leq f(x)$ for all x in $[a, b]$. We suppose that f and g are integrable on $[a, b]$, from which it follows that the function k defined by

$$k(x) = f(x) - g(x)$$

is also integrable on $[a, b]$. Let R be the region lying between the graphs of f and g over the interval $[a, b]$ and let A be the area of R. In other words, A is the area of the region of the plane bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. We begin with an approximation for A. First, we divide $[a, b]$ into n intervals of equal length

$$\Delta x = \frac{b - a}{n}$$

and let $a = x_0 < x_1 < x_2 < x_3 < \cdots < x_n = b$ be the endpoints of these intervals. Next, for $i = 1, 2, 3, \ldots, n$, let R_i be the region lying between the graphs of f and g over the

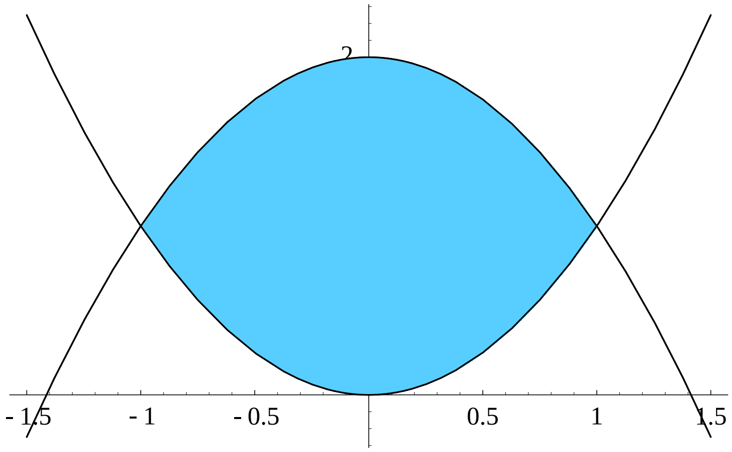

Figure 4.7.2 Region bounded by the graphs of $y = 2 - x^2$ and $y = x^2$

interval $[x_{i-1}, x_i]$. If A_i is the area of R_i, then

$$A = \sum_{i=1}^{n} A_i. \tag{4.7.1}$$

Now $f(x_i) - g(x_i)$ is the distance between the graphs of f and g at x_i, and so

$$(f(x_i) - g(x_i))\Delta x$$

should approximate A_i reasonably well when Δx is small. Thus

$$\sum_{i=1}^{n}(f(x_i) - g(x_i))\Delta x = \sum_{i=1}^{n} k(x_i)\Delta x \tag{4.7.2}$$

will approximate A. Moreover, we should expect that this approximation will improve as Δx decreases, that is, as n increases, and so we should have

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} k(x_i)\Delta x. \tag{4.7.3}$$

But now the right-hand side of (4.7.2) is a Riemann sum, in particular, the right-hand rule sum, and so the right-hand side of (4.7.3) converges to the definite integral of k on $[a, b]$. Hence we have

$$A = \int_a^b k(x)dx = \int_a^b (f(x) - g(x))dx. \tag{4.7.4}$$

Example Let R be the region bounded by the curves $y = 2 - x^2$ and $y = x^2$, as shown in Figure 4.7.2. Note that these curves intersect when

$$2 - x^2 = x^2,$$

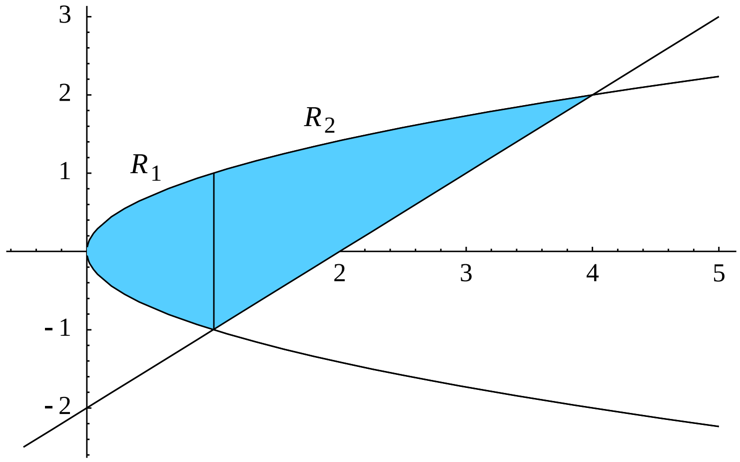

Figure 4.7.3 Region bounded by the graphs of $x = y^2$ and $x = y + 2$

which implies that $2x^2 = 2$, that is, $x = -1$ or $x = 1$. Hence the two curves intersect at $(-1, 1)$ and $(1, 1)$, and so we may describe R as the region between the curves $y = 2 - x^2$ and $y = x^2$ which lies above the interval $[-1, 1]$. Thus if A is the area of R, we have

$$\begin{aligned} A &= \int_{-1}^{1} ((2 - x^2) - x^2)dx \\ &= \int_{-1}^{1} (2 - 2x^2)dx \\ &= \left(2x - \frac{2}{3}x^3\right)\Big|_{-1}^{1} \\ &= \left(2 - \frac{2}{3}\right) - \left(-2 + \frac{2}{3}\right) \\ &= \frac{8}{3}. \end{aligned}$$

Example Let R be the region bounded by the curves $x = y^2$ and $x = y + 2$. These two curves intersect when

$$y^2 = y + 2,$$

which implies that

$$0 = y^2 - y - 2 = (y - 2)(y + 1).$$

Hence the two curves intersect when $y = -1$ and $y = 2$, that is, at the points $(1, -1)$ and $(4, 2)$. However, looking at Figure 4.7.3, we see that not all of R lies over the interval $[1, 4]$. In fact, R may be broken up into two regions, $R_1$and R_2, where R_1 is the region between the curves $y = \sqrt{x}$ and $y = -\sqrt{x}$ over the interval $[0, 1]$ and R_2 is the region between the curves $y = \sqrt{x}$ and $y = x - 2$ over the interval $[1, 4]$. Thus, if A is the area of R, A_1 is the area of R_1, and A_2 is the area of R_2, then

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_0^1 (\sqrt{x} - (-\sqrt{x})) + \int_1^4 (\sqrt{x} - (x-2))dx \\ &= \int_0^1 2\sqrt{x}\, dx + \int_1^4 (\sqrt{x} - x + 2)dx \\ &= \frac{4}{3}x^{\frac{3}{2}}\Big|_0^1 + \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{2}x^2 + 2x\right)\Big|_1^4 \\ &= \frac{4}{3} + \left(\frac{16}{3} - 8 + 8\right) - \left(\frac{2}{3} - \frac{1}{2} + 2\right) \\ &= \frac{9}{2}. \end{aligned}$$

The region R in the previous example may also be described as the region lying between the curves $x = y^2$ and $x = y+2$ over the interval $[-1, 2]$ on the y-axis. In general, analogous to our development above, if f and g are functions defined on an interval $[c, d]$ on the y-axis with $g(y) \leq f(y)$ for all y in $[c, d]$, then the area A of the region bounded by $x = f(y)$, $x = g(y)$, $y = c$, and $y = d$ (see Figure 4.7.4), is given by

$$A = \int_c^d (f(y) - g(y))dy. \qquad (4.7.5)$$

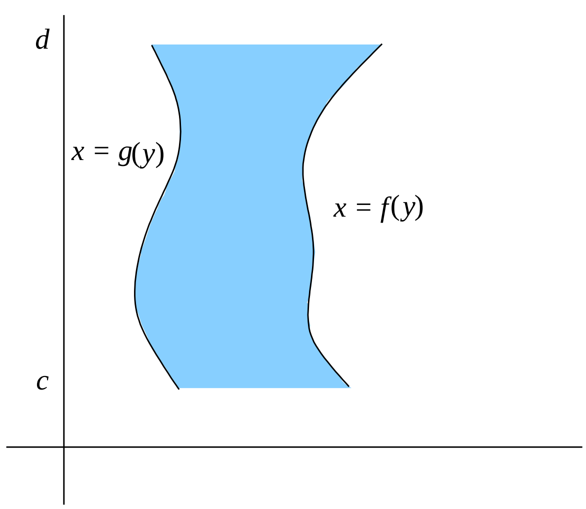

Figure 4.7.4 Region between the curves $x = f(y)$ and $x = g(y)$

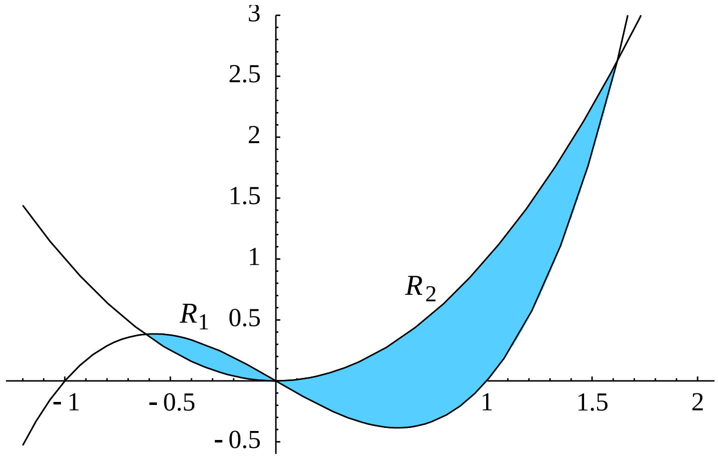

Figure 4.7.5 Region bounded by the graphs of $y = x^3 - x$ and $y = x^2$

In particular, for our previous example we have

$$\begin{aligned} A &= \int_{-1}^{2} (y + 2 - y^2)dy \\ &= \left(\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3\right)\Bigg|_{-1}^{2} \\ &= \left(2 + 4 - \frac{8}{3}\right) - \left(\frac{1}{2} - 2 + \frac{1}{3}\right) \\ &= \frac{9}{2}. \end{aligned}$$

In this case, the second method for solving the problem is a little simpler than the first; in general, it is often useful to look at a problem both ways and evaluate using the simpler of the two approaches.

Example Let R be the region bounded by the curves $y = x^3 - x$ and $y = x^2$. These curves intersect when

$$x^3 - x = x^2,$$

that is, when

$$0 = x^3 - x^2 - x = x(x^2 - x - 1).$$

Hence the curves intersect when $x = 0$,

$$x = \frac{1 - \sqrt{5}}{2},$$

or

$$x = \frac{1 + \sqrt{5}}{2},$$

where the latter two values were found using the quadratic formula. From the graphs in Figure 4.7.5, we see that R may be divided into two regions, R_1 and R_2, where R_1 extends from $x = \frac{1-\sqrt{5}}{2}$ to $x = 0$ and R_2 extends from $x = 0$ to $x = \frac{1+\sqrt{5}}{2}$. Note that in R_1 we have $x^3 - x \geq x^2$, whereas $x^2 \geq x^3 - x$ in R_2. Thus, if A is the area of R, A_1 is the area of R_1, and A_2 is the area of R_2, then

$$\begin{aligned} A &= A_1 + A_2 \\ &= \int_{\frac{1-\sqrt{5}}{2}}^{0} (x^3 - x - x^2)dx + \int_{0}^{\frac{1+\sqrt{5}}{2}} (x^2 - x^3 + x)dx \\ &= \left(\frac{1}{4}x^4 - \frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Bigg|_{\frac{1-\sqrt{5}}{2}}^{0} + \left(\frac{1}{3}x^3 - \frac{1}{4}x^4 - \frac{1}{2}x^2\right)\Bigg|_{0}^{\frac{1+\sqrt{5}}{2}} \\ &= \frac{13}{12}. \end{aligned}$$

Problems

1. Find the area of the region bounded by the curves $y = x$ and $y = x^2$.
2. Find the area of the region bounded by the curves $y = \sqrt{x}$ and $y = \dfrac{1}{2}x$.
3. Find the area of the region bounded by the curves $y = x^2$ and $y = x + 2$.
4. Find the area of the region bounded by the curves $y = \cos(x)$ and $y = x^2$.
5. Find the area of the region bounded by the curves $y = \sin(x)$ and $y = x^2$.
6. Find the area of one of the regions lying between the curves $y = \cos(x)$ and $y = \sin(x)$ between two consecutive points of intersection.
7. Find the area of the region in the first quadrant bounded by $y = \cos(x)$, $y = \sin(x)$, and $x = 0$.
8. Find the area of one of the regions lying between the curves $y = \cos^2(x)$ and $y = \sin^2(x)$ between two consecutive points of intersection.
9. Let R be the region bounded by the curves $y = x^2$ and $y = 2 - x$.
   (a) Set up an integral to find the area of R using functions of x.
   (b) Set up an integral to find the area of R using functions of y.
   (c) Evaluate the simpler of the integrals in (a) and (b).
10. Find the area of the region bounded by the curves $x = y^2 - 1$ and $x = 1 - y^2$.
11. Find the area of the region bounded by the curves $x = y^2$ and $x = 6 - y$.
12. Find the area of the region bounded by the curves $y = x^3 - 2x$ and $y = x^2$.
13. Find the area of the region bounded by the curves $y = x^4 - 4x^2$ and $y = 3x^3$.
14. To estimate the surface area of a lake, 21 measurements of the width of the lake are made at points spaced 50 yards apart from one end of the lake to the other. Suppose the measurements are, in order, 0, 50, 100, 120, 180, 240, 300, 250, 220, 295, 305, 265, 240, 275, 225, 180, 120, 90, 63, 40, and 0, all measured in yards. Use Simpson's rule to approximate the surface area of the lake.