**INTRODUCTION**

Lake Erie, the smallest by volume of the Great Lakes, has been plagued by toxic algal blooms covering a large portion of its surface for the last few decades. The toxins from these blooms are dangerous for the ecology in the lake and can build up to the point where they affect the metropolitan water supplies for cities on the lake (such as Toledo, Ohio). Prediction of actual algal blooms is very unsure, so rather than predict algal blooms, we predict where in the lake will have a phosphorus (P) concentration above recommended levels (1.5µg/L). To predict this, we do two simulations of phosphorus diffusion across Lake Erie’s Western Basin using the finite difference method for a point source diffusion and a uniform western edge.

**ASSUMPTIONS**

- Lake Erie is a still body of water and ions spread only through diffusion
- Algal blooms can be represented as a two-dimensional surface
- Lake Erie can be approximated as a 125 km by 70 km rectangle
- Lake Erie is uniformly its mean depth, 19m
- The Western basin the only place that Phosphorus will enter the system
- The Eastern basin is the only place through which Phosphorus will leave the system and remains at its initial condition

**PHOSPHATE INPUT**

Diffusion of P in water: 4km³/day

Total P input to Lake Erie: 11,000 metric tons = 1.1e11 mg

Western Basin Phosphorus Input: 3000 metric tons

We modeled our time dependent function of the entry of phosphorus into Lake Erie based off data collected from the Maumee River, one of the largest sources of phosphorus into the lake and, using Matlab’s curve fitting toolbox, found a five term Fourier fit approximation for this cyclical data.

**VISUAL REFERENCE**

**ANALYTIC SOLUTION – UNIFORM DIFFUSION**

By modeling a uniform western edge, we can extend this 1-D analysis to our 2-D system.

We have \( U(x, t) \) and

\[
U_i = \sum_{k=0}^{N} \sum_{n=0}^{N} U_{kn} e^{i(n \beta x + k \gamma t)}
\]

Where \( U_{kn} = \varphi_{kn}(x) \eta_{kn}(t) = \varphi_{kn} \eta_{kn} \)

We solve in the general homogeneous method, where \( \varphi(x, t) = \varphi(t) \), \( \varphi(x, t) = \varphi(t) \phi(x) \) and \( \psi(x, t) = \psi(t) \psi(x) \) and \( \psi(t) \) is the linear ‘steady state’ solution, and \( \varphi(t) \eta(t) \) is the ‘deviation’. However, because our boundary conditions vary with time, we use \( \psi(t) \varphi(t) \). Thus,

\[
\frac{\partial \varphi}{\partial t} - \varphi \frac{\partial \eta}{\partial x} = \frac{\partial \psi}{\partial x} \frac{\partial \eta}{\partial t}
\]

But we know \( \varphi \eta \) from initial conditions:

Putting into the original equation,

\[
\varphi \frac{\partial \psi}{\partial x} \eta + \varphi \psi \frac{\partial \eta}{\partial t} = \frac{\partial \psi}{\partial x} \frac{\partial \eta}{\partial t}
\]

Set \( \psi = \varphi \eta \), and realize \( \frac{\partial \psi}{\partial t} = \varphi \frac{\partial \eta}{\partial t} \), \( \frac{\partial \psi}{\partial x} = \frac{\partial \eta}{\partial t} \), \( \frac{\partial \eta}{\partial t} = \frac{\partial \psi}{\partial x} \psi \)

Solving, \( \psi \varphi \frac{\partial \psi}{\partial x} \varphi \psi \eta = \psi \varphi \eta \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} \psi \eta \)

Which leads, \( \nu \psi \frac{\partial \psi}{\partial x} \psi \varphi \eta = \psi \varphi \eta \frac{\nu}{\partial x} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} \psi \eta \)

Going back to \( \psi \frac{\partial \psi}{\partial x} \psi \varphi \eta = \psi \varphi \eta \frac{\nu}{\partial x} \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} \psi \eta \)

\[
U(x, t) = \hat{U}(x) = \hat{U} \phi(x)
\]

**NUMERICAL SOLUTIONS**

We ran two simulations for these models using these parameters in Matlab using the Finite Difference method of numerical simulation. The first was assuming phosphorus was diffused evenly along the lake’s western shore. The second assumed phosphorus was diffused from two point sources representing the Detroit and Maumee rivers.

**ACKNOWLEDGMENTS**

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References for our model include work done by Langtangen, Pierce, and Rap respectively.

**REFERENCES**

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**NUMERICAL - UNIFORM DIFFUSION**

Uniform Diffusion model after 9 days. Can clearly see the gradient across the lake from the phosphorus coming in.

Uniform Diffusion model in Western Basin after 149 days. The phosphorus inputs have greatly raised the level of phosphorus in the lake, and the periodic nature of the input can clearly be seen.

**NUMERICAL - POINT SOURCE**

Point Source Diffusion model after 2 days. Can clearly see the two sources and the curved gradient as the phosphorus travels along the northern and southern shores.

Point Source Diffusion model after 17 days. Curved gradient continues to be observed near the sources, but farther to the East, a linear gradient, similar to the uniform diffusion can be seen.

**CONCLUSION**

Our physical results suggest BAP levels widely above the EPA recommendations, as expected. We also observe that over time and space, the point source model grows more similar to the uniform diffusion model, validating the assumption of uniform diffusion. We do not claim that the numerical solutions are quantitatively precise due to numerical and parameter uncertainty.

In general, this model is an interesting experiment in simplification of complex systems. While not every complication of the Erie system was captured, and many deliberately laid aside, the model accurately describes Lake Erie’s broad qualitative seasonal chemical behaviors.