Simple Atmospheric Models and Chaos

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Overview

Natural systems are often nonperiodic and irregular. To model systems such as hydrodynamic flow, Edward Lorenz used a system of PDEs that represent the governing laws of hydrodynamical systems. However, particular nonperiodic solutions to these equations cannot always be determined except by numerical procedures. To do this, we convert the system of PDEs into a simplified system of ODEs and use the nonperiodic solutions to these equations to develop a simple model of atmospheric conditions.

Saltzman Convection Equations

The flow that happens in a layer of fluid with a depth of H will produce a steady state solution if the temperature of the upper and lower layers remain constant. If the temperatures differ or are not constant, the solution is unstable and convection will occur. When no variations in the direction of the y-axis occurs and motion is parallel to the x-z axis, then the governing equations of the system are:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial(\psi,\nabla^2\psi)}{\partial(x,z)} + v\nabla^2\psi + g \propto \frac{\partial\theta}{\partial x}$$

$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + k\nabla^2\theta$$

Where ψ is a stream function, θ is the departure of temperature from a state of no convection, g is acceleration due to gravity, ∝ is the thermal expansion coefficient, v is kinetic viscosity, and k is thermal conductivity. Certain fields of motion will form when the quantity $R_a = g\alpha H^3 \Delta T v^{-1} k^{-1}$, the Rayleigh number, exceeds the critical value $R_c = \pi^4 a^{-2}(1+a^2)^3$.

Lorenz Equations

The above system of PDEs give us a model for convection and when the solution is unstable, its solutions are non-periodic. In order to find particular solutions to this problem, we can convert the system of PDEs into a system of ODEs by expanding ψ and θ into a double Fourier series.

$$\psi(y,z,t) = \sum_{m,n \in Z} a_{n,m} e^{-in\pi z}\, e^{-im\pi y}$$

$$\theta(y,z,t) = \sum_{m,n \in Z} b_{n,m} e^{-in\pi z}\, e^{-im\pi y}$$

We then isolate these into a single term expansion and take the Jacobian matrix . Then, integrating the equations numerically returns finite solutions rather than infinite ones. Shortening the finite expressions gives us

$$a(1+a^2)^{-1}\psi = X\sqrt{2}\sin(\pi a H^{-1}X)\sin(\pi H^{-1}Z)$$
$$\pi R_a {R_c}^{-1} \Delta T^{-1}\theta = Y\sqrt{2}\cos(\pi a H^{-1}x)\sin(\pi H^{-1}z) - Z sin(2\pi H^{-1}z)$$

and substituting these expressions into the Saltzman equations returns a simplified model for convection currents where the variables x, y, and z are dependent on time alone.

$$X' = \sigma(Y - X)$$
$$Y' = X(r - Z) - Y$$
$$Z' = XY - bZ$$

Solving Lorenz Equations

Evaluating the general solution to this system of equations returns the characteristic equation:

$$\lambda^3 + (\sigma + b + 1)\lambda^2 + (r+\sigma)b\lambda + 2ab(r-1) = 0$$

We want our solutions to be nonperiodic and unstable. If r >1, the characteristic equation has two complex roots that are purely imaginary when $r = \sigma(\sigma+b+3)(\sigma-b-1)^{-1}$ and σ >b+1. To model this, we choose r =28, σ =10, b=8/3, and $x_0 = (1,1,1)$ for our initial conditions . The resulting solution returns a chaotic graph that varies greatly depending on initial conditions. Using (1,1,1) gives us results that vary from steady state initial conditions.

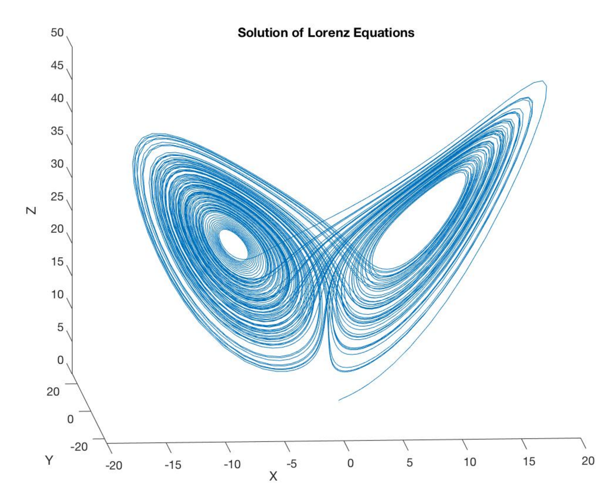

Figure 1

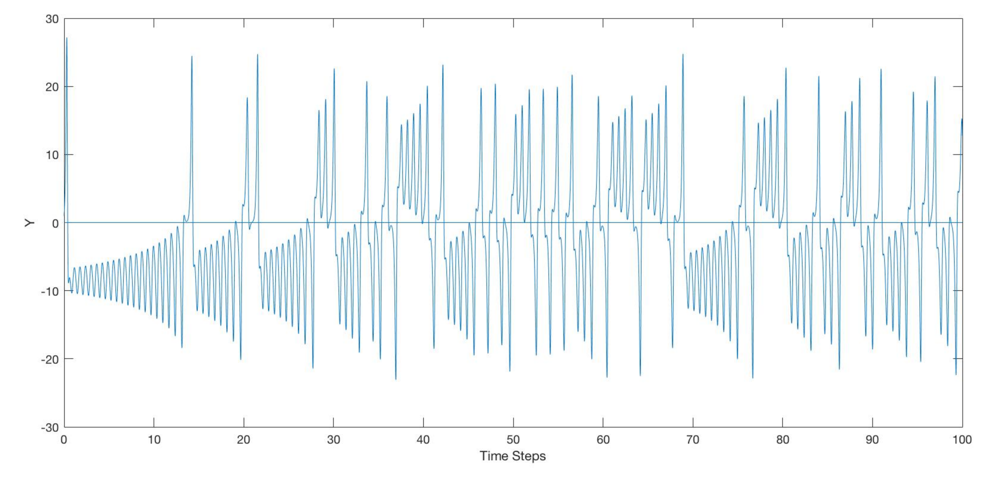

Figure 2

Figures: *Figure 1 is a model of the solution to the Lorenz system modeled in MATLAB with σ =10, b=8/3, and r=28. It is notable that the paths of the graph do not intersect themselves and that the origin (0,0,0) exists in a state of no convection while the two critical points exist in a steady state. Additionally, these two points are symmetric along the z-axis and called "Lorenz attractors". Figure 2 is a graph of the relationship between time and the y variable. The changes in maximums represent regime changes in weather systems.*

Modeling the Atmosphere Based on Initial Conditions

The solution above provides us with a 3 dimensional model of the system of Lorenz equations. This can be interpreted as a system where shifting from positive to negative y-values represents a weather regime change of different systems. When a regime change occurs, the result is a change in overall temperature and chances of precipitation.

The Rayleigh number r contains the most initial conditions of the atmosphere. The critical point of the Saltzman equation, when the system becomes nonperiodic, is R_c. The transformations used to arrive at the Lorenz equations, $r = \frac{R_a}{R_c}$.. By changing the initial conditions to reflect actual weather data in Hanover, NH for April 21, we may be able to predict atmospheric conditions for the next week. We keep the variable σ =10 the same and let a^2, which is representative of the wave number of convection rolls, equal 1/2 so that b=8/3. This gives us $R_c = \frac{27\pi^4}{4}$. Data from NOAA gives us the variables necessary to calculate R_a. Once we calculate R_a we find that $r = 37.38$. The variables x, y, z represent the intensity of convection in the system, the difference in temperature between the ascending and descending currents, and the deviation of the vertical temperature profile from linearity respectively. When x and y have the same signs, this indicates that warm air is rising and cold air is descending, which is how air behaves in our atmosphere. Based on this, we choose the initial conditions to be $x_0 = (.5,\ 2.3,\ -.8)$. The graph of x, y, z versus time is shown below.

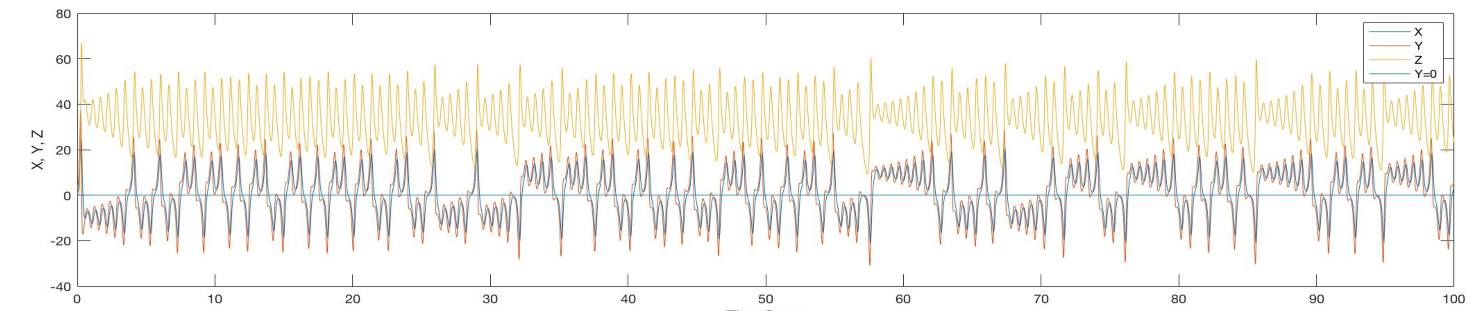

Figure 3: A graph of the x, y, and z, variables vs time, using 100 time steps in MATLAB. Initial conditions: σ =10 ; b=8/3; $r = 37.38$; $x_0 = (.5,\ 2.3,\ -.8)$

Now that we have this information, it is possible to predict future weather patterns by examining when the system completes a circuit around one of the attractors. We can do this using the maxima of z, which occur when a circuit around the Lorenz attractors is nearly complete. Variables x and y change signs once some critical value is reached in the z graph. In this case, it appears that this critical value occurs when the maxima of z is roughly 49. When this maxima is reached, it should represent a regime change in the system.

Regime Change Predictions April 21-30

Date	Type of Change	Approx. Time	Severity (1-10)
21	Warm	7:30AM	8
23	Cold	12:00PM	6
24	Warm	5:45AM	5
25	Cold	11:00AM	3
26	Warm	7:00AM	7
27	Cold	9:00AM	7
28	Warm	4:45PM	4
29	Cold	8:30AM	5

Figure 4: A table of predictions based on the above graph and maxima of z. Severity is determined based on how steep the slope of x and y is when they cross the .t-axis at 0.

Accuracy of Predictions

To see how accurate our model is at predicting the weather, we analyze trends in weather patterns in the Hanover area. We examine data such as, temperature, precipitation, pressure, wind, and humidity from the Lebanon Municipal Airport for our comparison. When a system transitions to a new regime, some of the effects are changes in average temperature, increased chances of precipitation, and slight increases or decreases in pressure. Significant drops or rises in these categories throughout indicate a change in the type of weather regime present. Small fluctuations throughout the day are reflected in the Lorenz plot as well when the x and y variables remain above or below 0. Compared to actual data, the model fairly accurately predicts transitions in weather patterns for the first few days, sometimes only missing changes by less than an hour. Our predictions start to become less reliable around the 25th, or roughly 4 days after the initial starting point. This is most likely due to slightly inaccurate initial conditions, the reason for which will be explored in the next section.

Chaotic Behavior

Lorenz systems are deterministic and nonperiodic. The initial conditions determine the behavior of the model, however, minute changes in initial conditions greatly impact the outcome of future behavior as time increases toward infinity. To show this, let us suppose the n^{th} maxima of z is equal to M_n. If we graph M_n versus M_{n+1}, we find that there is an order to the chaotic system. Indeed, our plot of M_n versus M_{n+1} shows that there is an approximate two to one relationship between these two values.

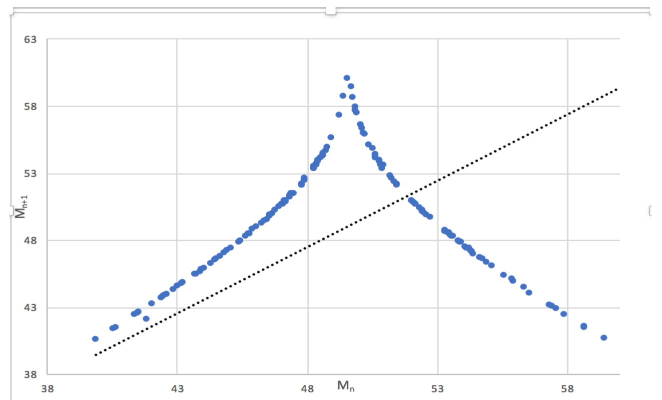

Figure 5: When we chart M_n versus M_{n+1}, we can see the order within the chaos. Knowing the n^{th} peak value will allow us to calculate the value of the peak n+1, but only for a short time. After this time, prediction becomes impossible as "chaos" in the system increases. The line y=x is added to the plot for comparison.

If we imagine an idealized form of the recursion, where the plot is a perfect 'tent' with a slope of 2, then the recursion is satisfied by $M_{n+1} = m_n \pm 2^n M_0$. Now, consider two sequences $M_0 \ldots M_n$ and $M_0' \ldots M_n'$ and suppose $M_0' = M_0 + \epsilon$. Then for small ϵ we have $M_n' = M_n \pm 2^n\epsilon$. What this relationship shows us is that small modifications result in unstable sequences and every one of these sequences is nonperiodic. However, because our results are based in simulation, this is not enough to formally prove these conclusions.

Conclusion

The Lorenz equations were instrumental in establishing the baseline for chaos theory, or what is known as "the butterfly effect". What this system shows us is that small changes in initial conditions greatly impact the outcome of solutions as time increases. These systems are still deterministic, but lose their insight after a relatively short amount of time. Predicting the weather based on this system is heavily dependent on the accuracy of the data collected prior to a forecast (in fact the initial conditions would have to be *exactly* accurate for predictions to remain valid). Slight deviations will eventually become incredibly significant as time passes. Approximating the present does not approximate the future, and perhaps the flapping of a butterfly's wings will drastically alter the course of events.

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