The Heat Equation of Flat Two- and Three-Dimensional Objects
Mia Dursht and Grace Hannam
Department of Mathematics, Dartmouth College

INTRODUCTION
The heat equation is a partial differential equation (PDE) that describes the distribution of heat over time. It was developed by Joseph Fourier in 1822 and has become crucial in many applications of physics and mathematics.
The general form of the heat equation is given by:
\[ u_t = k \nabla^2 u \]
where \( k \) is a positive constant
Using the two- and three-dimensional equivalents of this equation, this project aims to model the distribution of heat on the torus and the solid cube. It also explores potential methods for solving the heat equation of the Möbius strip.

METHOD: TORUS
To analyze the heat distribution on the torus, it is necessary to look at its 2D representation as an unfolded rectangle. Therefore, the heat equation in 2D, given by
\[ u_{tt} = k (u_{xx} + u_{yy}) \]
is used. From the rectangle, we obtain the Neumann boundary conditions:
\[ u(0,y,t) = u(L,y,t) , u(x,0,t) = u(x,J,t) \]
Assume solution of the form
\[ u(x,y,t) = f(x)g(y)h(t) \]
From the 2D heat equation, and by separation of variables, the equation becomes:
\[ k \left( \frac{f''(x)}{f(x)} + \frac{g''(y)}{g(y)} \right) = \lambda_{nm} = \lambda_n + \lambda_m \]
Evaluating each spatial variable individually to find its corresponding eigenvalue and eigenfunctions \( f(x) \) and \( g(y) \) allows for solving for \( h(t) \), the function in time. Once all three functions have been found, \( u(x,y,t) \) is written as a summation for all positive integers \( n,m \) of the product \( f(x)g(y)h(t) \).

RESULTS: TORUS
The general solution found is:
\[ u(x,y,t) = \sum_{n,m} a_{nm} \exp\left[-k \lambda_{nm}t\right] \cos\left(\frac{n\pi x}{L}\right) \cos\left(\frac{m\pi y}{J}\right) \cos\left(\frac{2\pi t}{T}\right) \]
where,
\[ \lambda_{nm} = -\left[ \frac{n^2 \pi^2}{L^2} + \frac{m^2 \pi^2}{J^2} \right] \]

ACKNOWLEDGMENTS AND REFERENCES:
(1) Professor Dorothy Wallace, Department of Mathematics, Dartmouth College
(4) https://gauss.math.yale.edu/~mr2245/pdespr2017Data/LecNotes.pdf
(5) https://physics.bgu.ac.il/~dcohen/ARCHIVE/mbs_PRB.pdf

METHOD: CUBE
Given a solid cube of volume \( V \) [see figure 3] that is insulated on the boundary \( dV \), it is necessary to use the heat equation in three dimensions,
\[ u_{tt} = k \nabla^2 u \]
to model its heat distribution. The insulated boundary gives Dirichlet boundary conditions:
\[ f'(0)=f'(L)=0, g'(0)=g'(J)=0, h'(0)=h'(K)=0 \]
Assume solution of the form
\[ \begin{align*}
  u(x,y,z,t) & = f(x)g(y)h(z)p(t) \\
  f(0) & = f(L) = 0, \\
  g(0) & = g(J) = 0, \\
  h(0) & = h(K) = 0 \\
\end{align*} \]
From the 3D heat equation, and by separation of variables, the equation becomes:
\[ k \left[ \frac{f''}{f} + \frac{g''}{g} + \frac{h''}{h} \right] = \lambda_{nmq} = \lambda_n + \lambda_m + \lambda_q \]
Evaluating each spatial variable individually to find its corresponding eigenvalue and eigenfunctions \( f(x) \), \( g(y) \), \( h(z) \) allows for solving for \( p(t) \), the function in time. \( u(x,y,z,t) \), the general solution for the heat distribution on the cube is the sum, for all positive integers \( n,m,q \), of the product \( f(x)g(y)h(z)p(t) \).

RESULTS: CUBE
The general solution found is, \( u(x,y,z,t) = \)
\[ \begin{align*}
  a_0 & + \sum_{n\neq 0} a_{nmq} \exp\left[-k \lambda_{nmq}t\right] \sin\left(\frac{2\pi n}{L}x\right) \sin\left(\frac{2\pi m}{J}y\right) \sin\left(\frac{2\pi q}{K}z\right) \\
\end{align*} \]
where,
\[ \lambda_{nmq} = -\left[ \frac{4\pi^2 n^2}{L^2} + \frac{4\pi^2 m^2}{J^2} + \frac{4\pi^2 q^2}{K^2} \right] \]

MÖBIUS STRIP
Treating the Möbius strip as a 2D shape [see figure 5], the boundary conditions that follow are:
\[ u(0,0,t) = u(L,J,t), u(0,J,t) = u(L,0,t) \]
However, neither boundary condition holds a spatial variable constant, and therefore, requires a different method of solving. Forcing Dirichlet boundary conditions \( f(0) = f'(L) = 0 \) allows for a solution for \( f(x) \). Assume
\[ g(y) = \sum_{n \neq 0} \frac{a_n}{\cos\left(\frac{n\pi J}{2}\right)} \cos\left(\frac{n\pi y}{J}\right) \]
However, evaluating \( f(x)g(y) = ku_t \) reveals \( f(x)g(y) \) is not a solution of the wave equation. Although no general solution was found, the Möbius strip is an interesting application of treating a 3D object in its 2D representation.